Stochastic Differential Equations under Nonlinear Mathematical Expectations and Applications

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Résumé en français

L'objectif principal de cette thèse est d'étudier certaines équations différentielles stochastiques (EDSs en abrégé) dans le cadre de la G-espérance, et certaines équations différentielles stochastiques rétrogrades (EDSRs en abrégé) du seconde ordre. Nous commençons par présenter quelques rappels sur les EDSRs, la G-espérance et les EDSRs du second ordre, ainsi que les principaux résultats obtenus dans le cadre de la G-espérance et pour les EDSRs du second ordre.

0.1 Rappels sur la théorie des EDSRs

0.1.1 Les EDSRs à croissance quadratique

Nous commençons par rappeler la formulation d'une équation différentielle stochastique rétrograde n-dimensionnelle :

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s^{\mathbb{P}_0},\ 0 \leq t \leq T,\ \mathbb{P}_0 - p.s., \tag{0.1}$$

où $(W_t^{\mathbb{P}_0})_{0\leq t\leq T}$ est un mouvement brownien d-dimensionnel sur un espace de probabilité complet $(\Omega, \mathcal{F}, \mathbb{P}_0)$, dont la filtration naturelle augmentée est notée $(\mathcal{F}_t)_{0\leq t\leq T}$. La valeur terminale ξ est une variable aléatoire $\mathcal{F}_T$-mesurable à valeurs dans $\mathbb{R}^n$ et le générateur $g : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ est une fonction progressivement mesurable par rapport à $(\mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n\times d}))_{0\leq t\leq T}$. Un couple de processus (Y, Z) est appelé solution de (0.1) si (Y, Z) est adapté par rapport à la filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ et vérifie (0.1).

Ce type d'équations a été introduit par Bismut [3] en 1973 dans le cas où le générateur g est linéaire, tandis que l'étude des EDSRs non-linéaires a pour origine l'article de Pardoux et Peng [68], dans lequel un théorème d'existence et d'unicité est donné dans le cas où g est uniformément lipschitzien en (y, z), $\xi \in L^2(\mathcal{F}_T; \mathbb{R})$ et

$$E^{\mathbb{P}_0}\left[\int_0^T |g(s,0,0)|^2 ds\right] < +\infty. \tag{0.2}$$

Depuis, cette théorie s'est considérablement développée. De nombreux auteurs ont travaillé à la recherche d'hypothèses plus faibles soit sur la valeur terminale, soit sur le générateur. Comme il serait trop long d'énumérer complètement tous ces travaux, nous introduirons uniquement quelques résultats pour les EDSRs quadratiques, qui sont étroitement liés à notre nouveau résultat sur les EDSRs du second ordre quadratiques dans cette partie.

Dans le cadre brownien, une des avancées les plus significatives pour la théorie des EDSRs est due à Kobylanski [46], qui a construit des solutions pour les EDSRs unidimensionnelles dont les générateurs sont à croissance quadratique en z, et les valeurs terminales sont bornées. Pour être plus précis, le générateur de ce type d'EDSRs satisfait l'hypothèse suivante : pour tout $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$|g(t, y, z)| \leq \alpha_t + \beta|y| + \frac{\gamma}{2}|z|^2,\ \text{uniformément en } (t, \omega), \tag{0.3}$$

où β et γ sont deux constantes positives, et α est un processus adapté positif satisfaisant la propriété d'intégrabilité suivante : il existe une constante positive C, telle que

$$\int_0^T \alpha_t dt \leq C,\ \mathbb{P}_0 - p.s..$$

Pour prouver l'existence d'une solution à (0.1) dans ce cadre, l'auteur de [46] construit une suite de solutions des EDSRs dont les générateurs vérifient une hypothèse quadratique et sont croissants (resp. décroissants) et minorées (resp. majorées) par une fonction linéaire en (y, z). De plus, si un changement de variable exponentiel est appliqué, ces EDSRs se transforment en des équations à coefficients lipschitziens. Par conséquent, l'existence de solutions pour ces EDSRs est assurée par le résultat de Pardoux et Peng [68], de plus on peut montrer que la suite de solutions est monotone par un principe de comparaison. Par une technique de convergence faible empruntée à d'EDPs, un théorème de stabilité monotone pour les EDSRs quadratiques est prouvé dans le même article [46] : il démontre que si la suite de Y converge uniformément sur $[0, T]$ trajectoire par trajectoire, la suite de Z converge dans $H^2(\mathbb{R}^d)$ pour la topologie forte. Grâce à ce théorème, une solution maximale (resp. minimale) à l'EDSR (0.1) peut être construite dans $S^\infty(\mathbb{R}) \times H^2(\mathbb{R}^d)$ comme limite de la suite énoncée ci-dessus. Nous présentons ensuite une version légèrement généralisée de ce théorème de stabilité monotone (cf. Briand et Hu [7]) :

Théorème 0.1 *Soient $\{\xi^n\}_{n\in\mathbb{N}}$ une suite de variables aléatoires $\mathcal{F}_T$-mesurables bornées, et $\{g^n\}_{n\in\mathbb{N}}$ une suite de générateurs continus en (y, z). Supposons que $\xi^n \to \xi$, $\mathbb{P}_0$-p.s., $g^n \to g$ localement uniformément en (y, z), et*

- $\sup_{n\in\mathbb{N}} ||\xi^n||_{L^\infty} < +\infty$ *;*
- $\sup_{n\in\mathbb{N}} |g^n(t, y, z)|$ *vérifie l'inégalité (0.3).*

Nous supposons de plus que pour chaque $n \in \mathbb{N}$, l'EDSR correspondant aux paramètres (ξ^n, g^n) admet une solution (Y^n, Z^n) dans $S^\infty(\mathbb{R}) \times H^2(\mathbb{R}^d)$, telle que la suite $\{Y^n\}_{n\in\mathbb{N}}$ est croissante (resp. décroissante).

Alors, la suite $\{Y^n\}_{n\in\mathbb{N}}$ converge vers $Y_t := \sup_{n\in\mathbb{N}} Y_t^n$ (resp. $\inf_{n\in\mathbb{N}} Y_t^n$) uniformément sur $[0, T]$, $\mathbb{P}_0$-p.s.. De plus, la suite $\{Z^n\}_{n\in\mathbb{N}}$ converge vers un certain Z dans $H^2(\mathbb{R}^d)$ et le couple (Y, Z) est une solution de l'EDSR correspondant aux paramètres (ξ, g).

Notons que le résultat d'existence obtenu par Kobylanski [46] a été amélioré : par exemple, Lepeltier et San Martín [48] fournissent un résultat dans le cas où la croissance du générateur g n'est plus linéaire en y ; Briand et Hu [7] considèrent les EDSRs dont les générateurs vérifient (0.3), mais dont les valeurs terminales ne sont plus bornées. Dans ces articles, l'approximation du générateur initial g est facile. Elle est donnée par :

$$g^n(t, y, z) := \sup_{(p,q)\in\mathbb{Q}^{1+d}} \{g(t, p, q) - n|p - y| - n|q - z|\}, \text{ pour chaque } n \in \mathbb{N}. \tag{0.4}$$

Si g vérifiant (0.3) est majorée par une fonction linéaire en (y, z), les générateurs g^n, $n \in \mathbb{N}$, de (0.4) sont lipschitziens, et la suite est décroissante. D'autre part, si g est minorée par une telle fonction, une suite croissante peut être également définie par inf-convolution. Dans le domaine des EDSRs, cette idée de construction par convolution a été issue de l'article de Lepeltier et San Martín [47] afin d'obtenir l'existence de solutions pour les EDSRs dont les générateurs sont continus et à croissance linéaire par rapport à y, mais lipschitziens en z.

Au sujet de l'unicité, Kobylanski [46] fournit un résultat sous certaines hypothèses techniques. Pourtant, en supposant que le générateur g est lipschitzien en y mais seulement localement lipschitzien en z, Hu et al. [34] prouvent que si la valeur terminale est bornée, Z est un générateur de martingale à oscillation moyenne bornée (martingale OMB en abrégé), et ainsi ils montrent l'unicité de la solution dans $S^\infty(\mathbb{R}) \times H^2_{BMO}(\mathbb{R}^d)$. L'hypothèse énoncée ci-dessus est de la forme suivante : pour tout $(y, z, z') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ et une certaine constante $K > 0$,

$$|g(t, y, z) - g(t, y, z')| \leq K|z - z'|(1 + |z| + |z'|), \text{ uniformément en } (\omega, t, y). \tag{0.5}$$

Par ailleurs, Briand et Hu [8] et Delbean et al. [16] traitent le même sujet, mais dans le cas où le générateur est convexe et la valeur terminale est non-bornée.

Signalons que les EDSRs à croissance quadratique en z sont utiles pour la résolution de problème de maximisation sous contraintes de l'utilité d'un portefeuille en finance, qui est de la forme suivante :

$$V(x) := \sup_{\pi\in\tilde{\mathcal{A}}} E^{\mathbb{P}_0}[U(X_T^{x,\pi})]. \tag{0.6}$$

Le premier résultat pour ce problème via la technique des EDSRs quadratiques est obtenu par El Karoui et Rouge [23], lorsque la fonction d'utilité est exponentielle et est donnée par $U(x) := -c(\exp(-x))$, $c > 0$, et la contrainte est convexe. Dans l'article [23], un problème dual du pricing est établi et la résolution de ce problème dual est donnée par la solution d'une EDSR quadratique.

Ce résultat est amélioré dans l'article de Hu et al. [34], où le problème initial est directement traité sans hypothèse de convexité sur la contrainte. Même si la méthode de Hu et al. [34] est adaptée pour le cas des utilités logarithme et puissance, nous allons nous placer dans le cardre d'une utilité exponentielle.

Considérons qu'il y a une seule obligation et n actifs sur le marché financier, $n \leq d$. Le taux d'intérêt de cette obligation est zéro et les processus de prix des actifs suivent les EDSs suivantes : pour un certain processus b borné et un certain processus σ tel que $\mathrm{rg}(\sigma) = n$ et $\sigma\sigma^{\mathbf{Tr}}$ est uniformément elliptique,

$$dS_t^i = S_t^i(b_t^i dt + \sigma_t^i dW_t^{\mathbb{P}_0}),\ 0 \leq t \leq T,\ \mathbb{P}_0 - p.s.,\ i = 1, \dots, n. \tag{0.7}$$

Soit $\tilde{\mathcal{A}}$ un ensemble de processus π, qui sont $\mathcal{F}$-progressivement mesurables, à valeurs dans une contrainte $\tilde{C}$, tels que $\int_0^T |\pi_t\sigma_t|^2 dt < +\infty$, $\mathbb{P}_0$-p.s., et $\{\exp(-cX_\tau^\pi)\}_{\tau \in \mathcal{T}_0^T}$ une famille uniformément intégrable. Ici, π désigne une stratégie de l'investisseur, où π_t^i désigne le montant investi dans l'actif i au temps t. Alors, la fonction de valeur s'écrit :

$$V(x) := \sup_{\pi \in \tilde{\mathcal{A}}} E^{\mathbb{P}_0}\left[- \exp\left(- c\left(x + \int_0^T \pi_t(\sigma_t dW_t^{\mathbb{P}_0} + b_t dt) - \xi \right)\right)\right],$$

où la variable aléatoire $\mathcal{F}_T$-mesurable et bornée ξ désigne un actif contingent autre que le portefeuille, qui arrive à la date T. Le but de l'investisseur est de choisir une des meilleures stratégies π appartenant à $\tilde{\mathcal{A}}$ qui optimise l'utilité espérée au temps T, i.e. la valeur de V. Dans l'article [34], une résolution de ce problème est obtenue grâce à la solution d'une EDSR quadratique. Plus précisément, la fonction de valeur peut être représentée par

$$V(x) = -\exp(-c(x - Y_0)),$$

où Y_0 est la solution de l'EDSR donnée par

$$Y_t = \xi + \int_t^T f(s, Z_s)ds - \int_t^T Z_s dW_s^{\mathbb{P}_0},\ 0 \leq t \leq T,\ \mathbb{P}_0 - p.s., \tag{0.8}$$

et où pour chaque $(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$f(\omega, t, z) := \frac{c}{2}\,\mathrm{dist}^2\left(z + \frac{1}{c}\theta_t(\omega), \tilde{C}\sigma_t(\omega)\right) - z^{\mathbf{Tr}}\theta_t(\omega) - \frac{1}{2c}|\theta_t(\omega)|^2.$$

avec $\theta_t := \sigma_t^{\mathbf{Tr}}(\sigma_t\sigma_t^{\mathbf{Tr}})^{-1}b_t$. Un calcul simple montre que f vérifie (0.3), (0.5) et les conditions usuelles pour les générateurs d'EDSRs. L'existence et l'unicité de la solution sont alors assurées pour (0.8).

Par ailleurs, les EDSRs quadratiques ont été étudiées dans un cadre non-brownien (citons par exemple, Morlais [64]), et aussi dans le cadre d'une filtration discontinue (voir Morlais [65, 66]).

0.1.2 La théorie de la g-espérance

En 1952, lorsque le paradoxe d'Allais a été mis en évidence, les économistes découvrent que la théorie de « l'utilité espérée » à base d'une espérance mathématique linéaire est contestable. L'intérêt pour une notion d'espérance mathématique non-linéaire se développe alors considérablement. Se pose alors une question : pouvons-nous trouver une nouvelle notion qui peut être une généralisation naturelle de l'espérance linéaire ? Notamment en préservant, autant que possible, les propriétés de l'espérance linéaire. Comme réponse à cette question, Peng propose dans l'article [71] une espérance non-linéaire, dite g-espérance, par la solution d'EDSRs.

Définition 0.2 *Soit g un générateur d'EDSR uniformément lipschitzien en y et z, et vérifiant (0.2) et $g(\cdot, \cdot, 0) \equiv 0$. Pour chaque $\xi \in L^2(\mathcal{F}_T; \mathbb{R})$, $Y(\xi, g)$ designe la solution de l'EDSR (0.1) correspondant aux paramètres (ξ, g). La g-espérance de ξ est définie par $\mathcal{E}^g[\xi] = \mathcal{E}_{0,T}^g[\xi] := Y_0(\xi, g)$, et la g-espérance conditionnelle correspondante par $\mathcal{E}^g[\xi|\mathcal{F}_t] = \mathcal{E}_{t,T}^g[\xi] := Y_t(\xi, g)$.*

La définition ci-dessus montre que la notion de g-espérance est dynamique, et cette nouvelle espérance non-linéaire est cohérente avec la filtration (voir [13] Coquet et al. pour la définition), ce qui est une différence significative entre cette notion d'espérance non-linéaire et les autres existantes.

Notons que $\xi \equiv Y_T$, par abus de terminologie, nous appelons alors sY une g-martingale, i.e. une martingale sous l'espérance non-linéaire $\mathcal{E}^g[\cdot]$. Comme extension naturelle de cette notion, Peng [72] définit les g-surmartingales et les g-sousmartingales :

Définition 0.3 *Supposons g vérifiant les mêmes conditions que dans la définition précédente mis à part $g(\cdot,\cdot,0) \equiv 0$ et Y un processus progressivement mesurable par rapport à $\mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}^n)$, à valeurs dans $\mathbb{R}$ et vérifiant pour chaque $t \in [0,T]$, $E[|Y_t|^2] < +\infty$. Alors, l'EDSR (0.1) correspondant aux paramètres (Y_t, g), dont le temps terminal est t, admet une unique solution y^t sur $[0,t]$. Si pour chaque $t \in [0,T]$, $y_s^t \leq Y_s$ (resp. $y_s^t \geq Y_s$), $\mathbb{P}_0$-p.s., pour tout $s \in [0,t]$, Y est appelée une g-surmartingale (resp. g-sousmartingale) dans un sens faible.*

Dans la définition ci-dessus, si tous les temps déterministes sont remplacés par des temps d'arrêt, nous disons que Y est une g-surmartingale (resp. g-sousmartingale) au sens fort.

À partir de la définition ci-dessus, Peng [72] fournit un théorème de décomposition de type Doob-Meyer pour les g-surmartingales (resp. g-sousmartingales) càdlàg. Ce résultat est généralisé dans l'article de Ma et Yao [59] pour les générateurs g vérifiant (0.3) et certaines hypothèses de Kobylanski [46], afin d'assurer l'unicités de solutions pour les EDSRs associées. Remarquons que si les hypothèses de Kobylanski [46] sont remplacées par l'hypothèse que g est lipschitzien en y et vérifie (0.5), le théorème de décomposition de Ma et Yao [59] est encore vrai. Dans la suite, nous présentons ce théorème modifié, qui joue un rôle très important pour prouver l'existence de solutions pour les EDSRs du second ordre quadratique dans le chapitre 4 :

Théorème 0.4 *Supposons que Y est un processus càdlàg, $\mathcal{F}$-adapté et borné, et que g est un générateur d'une EDSR, qui vérifie les conditions suivantes : (0.3), lipschitzien en y et (0.5). Alors, si Y est une g-surmartingale, il existe un processus càdlàg croissant K nul en 0 et un processus $Z \in H^2(\mathbb{R}^d)$, tel que*

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \leq t \leq T,\ \mathbb{P}_0 - p.s..$$

Nous considérons à nouveau la g-espérance $\mathcal{E}^g[\cdot]$ donnée par la définition 0.2, et introduisons la mesure de risque associée.

Nous définissons une fonctionnelle sur $L^2(\mathcal{F}_T;\mathbb{R})$ par la g-espérance $\rho(X) := \mathcal{E}^g[-X]$, et nous voyons que cette fonctionnelle est bien adaptée à la définition d'une mesure de risque statique de l'article d'Artzner et al. [2]. De plus, la g-espérance étant dynamique et cohérente avec la filtration, elle peut également générer une mesure de risque dynamique.

Dans l'article [2], une notion de mesure de risque cohérente est aussi établie. Pour le cas statique, elle est donnée par :

Définition 0.5 *Nous disons qu'une fonctionnelle ρ sur un espace linéaire $\mathcal{L}$ est une mesure de risque statique cohérente si les quatre axiomes suivants sont satisfaits : pour chaque X et $Y \in \mathcal{L}$,*

(1) ***Monotonie :*** *si $X \leq Y$, $\rho(X) \geq \rho(Y)$;*
(2) ***Sous-additivé :*** *$\rho(X+Y) \leq \rho(X) + \rho(Y)$;*
(3) ***Homogénéité positive :*** *$\rho(\lambda X) = \lambda\rho(X)$, pour toute constante $\lambda \geq 0$;*
(4) ***Invariance par translation :*** *$\rho(X+c) = \rho(X) - c$, pour toute constante $c \in \mathbb{R}$.*

Selon les résultats connus pour les EDSRs lipschitziennes, si le générateur g est indépendant de y, positivement homogène et sous-additive en z, la mesure de risque statique induite par $\mathcal{E}^g[\cdot]$ est cohérente. Sous les mêmes conditions sur g, nous avons une conclusion similaire dans le cas dynamique, i.e., la mesure de risque dynamique introduite par la g-espérance conditionnelle $\mathcal{E}^g[\cdot|\mathcal{F}_t]$ est aussi cohérente.

Pour plus de détails concernant la g-espérance, ses applications en finance et les connaissances de mesure de risque cohérente, nous renvoyons le lecteur aux articles de Artzner et al. [2], Chen et Epstein [9], Coquet et. al. [13], Ma et Yao [59], Peng [71, 72] et Rosazza [82].

0.2 Une étude dans le cadre de la G-espérance

0.2.1 Rappels du cadre de la G-espérance

Récemment, une nouvelle notion d'espérance sous-linéaire a été donnée par Peng [73] à l'aide d'un point de vue d'analyse fonctionnelle, sans considérer une EDSR sous-jacente. Nous expliquons cette idée dans la suite.

Nous désignons par $C_{l,Lip}(\mathbb{R}^n)$ l'ensemble des fonctions localement lipschitziennes sur $\mathbb{R}^n$ et par $\mathcal{H}$ un réseau de vecteurs composé de fonctions réelles définies sur un ensemble Ω et fermé par rapport à $C_{l,Lip}(\mathbb{R}^n)$ pour chaque $n \in \mathbb{N}^+$, i.e. : pour tout $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ et $X_1, \ldots, X_n \in \mathcal{H}$, $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$. Cet ensemble $\mathcal{H}$ peut être vue comme un espace de positions admissibles en finance. Nous disons qu'une fonctionnelle $\hat{\mathbb{E}}[\cdot]$ sur $\mathcal{H}$ est une espérance sous-linéaire, si la mesure de risque associée $\rho(X) := \hat{\mathbb{E}}[-X]$, $X \in \mathcal{H}$, est cohérente, autrement dit si $\rho(\cdot)$ possède les quatre propriétés données dans la définition 0.5. Dans ce nouveau contexte, nous appelons le triplet $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ un espace d'espérance sous-linéaire, et $X \in \mathcal{H}$ une variable aléatoire.

Parallèlement aux concepts du cadre classique, Peng [74] établit les notions de distribution et d'indépendance pour les variables aléatoires dans ce nouveau contexte. Néanmoins, ces notions sont moins probabilistes mais plutôt fonctionnelles, et elles s'expriment à l'aide des familles de fonctions tests $C_{l,Lip}(\mathbb{R}^n)$, $n \in \mathbb{N}^+$.

Définition 0.6 *Soit X un vecteur aléatoire $(X_1, \ldots, X_n)$, où $X_i \in \mathcal{H}$, $i = 1, \ldots, n$. La distribution de X est donnée par la fonctionnelle $\mathbb{F}_X[\cdot]$ suivante : pour chaque $\varphi \in C_{l,Lip}(\mathbb{R}^n)$,*

$$\mathbb{F}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)].$$

De plus, si X' est un autre vecteur aléatoire n-dimensionnel et pour chaque $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, $\mathbb{F}_X[\varphi] = \mathbb{F}_{X'}[\varphi]$, X et X' sont dits identiquement distribués.

Soit $Y := (Y_1, \ldots, Y_m)$, où $Y_i \in \mathcal{H}$, $i = 1, \ldots, m$. Le vecteur Y est dit indépendant de X si pour chaque $\varphi \in C_{l,Lip}(\mathbb{R}^{n+m})$,

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Nous remarquons que ces nouvelles définitions sont compatibles avec celles du cadre classique si l'espérance $\hat{\mathbb{E}}[\cdot]$ est linéaire. Signalons cependant que cette relation d'indépendance n'est pas symétrique.

Il est bien connu que le théorème central limite et la loi des grands nombres sont des résultats fondamentaux dans la théorie classique en probabilités et statistique, qui expliquent de manière convaincante pourquoi la loi normale est couramment utilisée. Remarquons qu'un théorème et une loi similaires à ces derniers sont établis par Peng dans ce nouveau contexte.

Nous considérons une suite de variables aléatoires d-dimensionnelles $\{X_i\}_{i \in \mathbb{N}^+}$, telle que $\hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0$, toutes les X_i sont identiquement distribuées et pour chaque $i \in \mathbb{N}^+$, X_i est indépendante de $(X_1, \ldots, X_{i-1})$. Peng [74] démontre que la suite $\{\bar{S}_n\}_{n \in \mathbb{N}^+}$ donnée par

$$\bar{S}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i,$$

converge en loi vers une variable aléatoire X dont la loi est caractérisée par

$$\hat{\mathbb{E}}[\varphi(X)] = u^\varphi(1, 0), \ \varphi \in C_{l,Lip}(\mathbb{R}^d),$$

où u^φ est la solution de viscosité de l'EDP parabolique suivante sur $\mathbb{R}^+ \times \mathbb{R}^d$, appelée équation de Black-Scholes-Barenblatt :

$$\begin{cases} \dfrac{\partial u}{\partial t} - G(D^2 u) = 0; \\ u|_{t=0} = \varphi, \end{cases}$$

et dont le noyau de chaleur $G : \mathbb{S}^d \to \mathbb{R}$ est défini par

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[(AX_1, X_1)], \tag{0.9}$$

et où $\mathbb{S}^d$ est l'ensemble des matrices symétriques réelles d'ordre d et on utilise l'opérateur $(\cdot,\cdot)$ pour désigner le produit scalaire. Cette loi est associée à la fonction G donnée en (0.9), ainsi elle est dite loi G-normale sous l'espérance sous-linéaire $\hat{\mathbb{E}}[\cdot]$ et ce résultat est l'analogue du théorème central limite du cadre classique.

D'après Peng [74], il existe un sous-ensemble borné, convexe et fermé Γ de $\mathbb{R}^{d\times d}$, tel que pour chaque $A \in \mathbb{S}^d$, $G(A)$ peut-être représenté par

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Gamma} \mathrm{tr}[\gamma\gamma^{\mathbf{Tr}}A], \tag{0.10}$$

et la loi G-normale est alors notée par $\mathcal{N}(0,\Sigma)$, où $\Sigma := \{\gamma\gamma^{\mathbf{Tr}} : \gamma \in \Gamma\}$. En fait, cet ensemble Σ caractérise le fait que la variance de X est incertaine, et nous savons que si l'espérance $\mathbb{E}[\cdot]$ est linéaire, cet ensemble n'est composé que d'une seule matrice qui est la matrice de variance-covariance d'une variable aléatoire classique de loi normale.

Par ailleurs, la théorie des EDPs (cf. Crandall et al. [14]) assure l'existence et l'unicité de la solution de viscosité pour une telle EDP dont le noyau G est de la forme (0.10), donc l'existence d'une variable aléatoire de loi G-normale peut être bien prouvée (cf. Peng [74, 75]).

Sous l'espérance sous-linéaire énoncée ci-dessus, une loi des grands nombres est aussi obtenue, et la loi limite est appelée « loi maximale » et implique une moyenne incertaine.

Jusque-là, l'espérance sous-linéaire introduite est encore statique. Dans la suite, nous introduisons une formulation d'une espérance non-linéaire dynamique et cohérente à une filtration, dite la G-espérance.

Nous précisons maintenant Ω : il s'agit de l'espace des fonctions continues sur $\mathbb{R}^+$, à valeurs dans $\mathbb{R}^d$, nulles en 0, et muni de la distance suivante :

$$\rho(\omega^1,\omega^2) := \sum_{N=1}^{\infty} 2^{-N}((\max_{t\in[0,N]} |\omega_t^1 - \omega_t^2|) \wedge 1).$$

Pour T fixé, nous désignons par Ω_T l'ensemble des fonctions de Ω tronquées au temps T et par $L_{ip}^0(\Omega_T)$ l'espace des variables aléatoires de cylindre de dimension finie, donné par

$$L_{ip}^0(\Omega_T) := \{\varphi(B_{t_1},\dots,B_{t_n}) : n \geq 1,\ 0 \leq t_1 \leq \ldots \leq t_n \leq T,\ \varphi \in C_{l,Lip}(\mathbb{R}^{d\times n})\},$$

où B est le processus canonique. Considérons un espace d'espérance sous-linéaire $(\hat{\Omega},\hat{\mathcal{H}},\hat{\mathbb{E}})$ et une suite de variables aléatoires $\{\xi_i\}_{i\in\mathbb{N}}$ vérifiant que pour tout $i \in \mathbb{N}^+$, ξ_i suit une loi G-normale donnée et est indépendante de $(\xi_1,\dots,\xi_{i-1})$. Nous construisons une espérance sous-linéaire sur $L_{ip}^0(\Omega_T)$ de la façon suivante : pour chaque $X \in L_{ip}^0(\Omega_T)$ de la forme

$$X = \varphi(B_{t_1} - B_{t_0},\dots,B_{t_n} - B_{t_{n-1}}), \tag{0.11}$$

où $\varphi \in C_{l,Lip}(\mathbb{R}^{d\times n})$ et $0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$, nous définissons

$$\mathbb{E}[\varphi(B_{t_1} - B_{t_0},\dots,B_{t_n} - B_{t_{n-1}})] := \hat{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1,\dots,\sqrt{t_n - t_{n-1}}\xi_n)].$$

Il est facile de voir que $\mathbb{E}[\cdot]$ est une espérance sous-linéaire et sous $\mathbb{E}[\cdot]$, le processus canonique est un G-mouvement brownien, dans le sens suivant :

Définition 0.7 *X est appelé G-mouvement brownien d-dimensionnel sous une espérance sous-linéaire $\mathbb{E}[\cdot]$ si, pour tout $0 \leq s \leq t \leq T$, l'accroissement $X_t - X_s \sim \mathcal{N}(0,(t-s)\Sigma)$ et pour tout $n \in \mathbb{N}^+$ et $0 \leq t_1 \leq \ldots \leq t_n \leq T$, $X_{t_n} - X_{t_{n-1}}$ est indépendant de $(X_{t_1},\dots,X_{t_{n-1}})$. Une espérance sous-linéaire $\mathbb{E}[\cdot]$ sur $L_{ip}^0(\Omega_T)$ est appelée G-espérance, si le processus canonique B est un G-mouvement brownien sous $\mathbb{E}[\cdot]$.*

Pour chaque $p \geq 1$, nous pouvons définir une norme $\mathbb{E}[|\cdot|^p]^{1/p}$ sur $L_{ip}^0(\Omega_T)$ et nous désignons par $L_G^p(\Omega_T)$ le complété de $L_{ip}^0(\Omega_T)$ pour cette norme. Ainsi, la G-espérance est prolongée naturellement à ce complété.

Pour chaque $X \in L_{ip}^0(\Omega_T)$ de la forme (0.11), Peng définit également son espérance conditionnelle par rapport à la « filtration » $\{\Omega_t\}_{0\leq t\leq T}$ par :

$$\begin{aligned}\mathbb{E}[\varphi(B_{t_1} - B_{t_0},\dots,B_{t_j} - B_{t_{j-1}},B_{t_{j+1}} - B_{t_j},\dots,B_{t_n} - B_{t_{n-1}})|\Omega_{t_j}]\\ := \mathbb{E}[\psi(B_{t_1} - B_{t_0},\dots,B_{t_j} - B_{t_{j-1}})],\end{aligned}$$

où $\psi(x_1,\ldots,x_j) := \mathbb{E}[\varphi(x_1,\ldots,x_j,\sqrt{t_{j+1}-t_j}\xi_{j+1},\ldots,\sqrt{t_n-t_{n-1}}\xi_n)]$. Notons que pour un t n'appartenant pas à la subdivision $(t_0,\ldots,t_n)$, nous pouvons réécrire X par rapport à une nouvelle subdivision $(\hat{t}_0,\ldots,\hat{t}_j,\ldots\hat{t}_{n+1})$ pour avec un certain $\hat{t}_j = t$, ainsi, la G-espérance conditionnelle est définie sur $L^0_{ip}(\Omega_T)$ puis peut bien être prolongée à l'espace $L^p_G(\Omega_T)$.

En utilisant le théorème de Hahn-Banach, Peng [74] montre qu'une espérance sous-linéaire peut être représentée par une espérance supérieure d'une classe d'espérances linéaires. Mais une question se pose alors : pouvons-nous construire explicitement cette classe ? Signalons qu'auparavant, l'idée d'espérance supérieure a été fournie par Huber [40] dans un contexte de statistique robuste, et que ce théorème de représentation similaire à celui de Peng [74] a été prouvé dans l'article de Delbaen [15], mais pour les mesures de risque cohérentes.

Afin de résoudre le problème de pricing de créances conditionnelles dans un contexte de modèles incertains, Denis et Martini [19] développent une technique d'analyse stochastique quasi-sûre se basant sur la théorie de la capacité. Dans cet article, la capacité est définie par une espérance supérieure d'une classe d'espérances linéaires générées par certaines mesures de martingale. Dans l'article de Denis et al. [17], les auteurs remarquent que le cadre de Denis et Martini [19] est étroitement lié à celui de la G-espérance proposée par Peng [73, 74], et que la G-espérance peut donc être construite de cette façon.

Nous désignons par $\mathbb{P}_0$ la mesure de Wiener, et il est connu que le processus canonique B est un mouvement brownien sous $\mathbb{P}_0$. Nous construisons ensuite une collection $\mathcal{A}^{\Gamma}_{[0,+\infty)}$ composée des processus d-dimensionnels θ qui sont $\mathcal{F}^B$-progressivement mesurables et à valeurs dans Γ. Pour chaque $\theta \in \mathcal{A}^{\Gamma}_{[0,+\infty)}$, considérons la probabilité $\mathbb{P}_\theta$ induite par la formulation forte suivante :

$$\mathbb{P}_\theta := \mathbb{P}_0 \circ (X_\theta)^{-1},$$

où $X_\theta := (\int_0^t \theta_s dB_s)_{t\geq 0}$, $\mathbb{P}_0$-p.s.. Notons $\mathcal{P} := \{\mathbb{P}_\theta : \theta \in \mathcal{A}^{\Gamma}_{[0,+\infty)}\}$. Selon Denis et al. [17], cet ensemble $\mathcal{P}$ de probabilités est tendu, et son adhérence $\mathcal{P}_G$ pour la topologie faible est alors compacte. De plus, la G-espérance $\mathbb{E}[\cdot]$ coïncide sur $L^1_G(\Omega_T)$ avec l'espérance supérieure suivante :

$$\bar{\mathbb{E}}[X] := \sup_{\mathbb{P}\in\mathcal{P}_G} E^{\mathbb{P}}[X]. \tag{0.12}$$

De cette manière, la G-espérance $\mathbb{E}[\cdot]$ est bien prolongée par (0.12) de $L^1_G(\Omega_T)$ à l'ensemble des variables aléatoires $\mathcal{B}(\Omega_T)$-mesurables. Dans la suite, $\bar{\mathbb{E}}[\cdot]$ est vue comme une G-espérance mais définie pour toute $X \in \mathcal{B}(\Omega_T)$, et nous ne distinguons plus les deux notations $\mathbb{E}[\cdot]$ et $\bar{\mathbb{E}}[\cdot]$.

Dans cet ensemble $\mathcal{P}$, toutes les mesures de probabilité sont mutuellement singulières : nous ne pouvons plus trouver une mesure finie qui domine toute $\mathbb{P} \in \mathcal{P}$, ce qui fait de la G-espérance un outil adapté pour la résolution d'un problème possédant des modèles non-dominés. Néanmoins, ce type de structures $\mathcal{P}$ nous présente certaines difficultés : Denis et al. [17] fournissent un théorème de convergence monotone pour une suite décroissante dont les variables aléatoires appartiennent à $L^1_G(\Omega_T)$, mais il manque un théorème plus général ; ainsi, dans le cadre de la G-espérance, nous n'avons plus comme dans le cadre classique un théorème de convergence dominée, ce qui pose des problèmes lorsque nous nous plaçons dans ce cadre.

En considérant la forme équivalente (0.12) de la G-espérance, nous savons que l'espace $L^1_G(\Omega_T)$ est plus petit que le complété de $L^0_{ip}(\Omega_T)$ pour la norme $E^{\mathbb{P}}[|\cdot|]$, pour une certaine $\mathbb{P} \in \mathcal{P}$, parce que $\bar{\mathbb{E}}[|\cdot|]$ est plus forte. Désignons par $C_b(\Omega_T)$ l'espace des variables aléatoires continues (en ω) et bornées, et par $B_b(\Omega_T)$ l'espace des variables aléatoires $\mathcal{B}(\Omega_T)$-mesurables et bornées. Il est facile de voir que pour une certaine norme $E^{\mathbb{P}}[|\cdot|]$, les complétés de $C_b(\Omega_T)$ et $B_b(\Omega_T)$ sont identiques à celui de $L^0_{ip}(\Omega_T)$. Par contre, dans le nouveau contexte, selon Denis et al. [17], le complété de $B_b(\Omega_T)$ est plus grand que les complétés de $L^0_{ip}(\Omega_T)$ et $C_b(\Omega_T)$ pour la norme $\bar{\mathbb{E}}[|\cdot|]$. Les deux derniers sont équivalents, car $L^0_{ip}(\Omega_T)$ est dense dans $C_b(\Omega_T)$ pour la topologie uniforme.

Dans le cadre classique, l'intégrale d'Itô est définie tout d'abord sur un espace de processus simples par des sommes de Riemann-Stieltjes, ensuite, cette définition est étendue aux processus intégrables pour $E^{\mathbb{P}}[\int_0^T |\cdot|^2 dt]$. Pour définir l'intégrale de G-Itô, on répète cette construction dans le cadre de G-espérance. Dans les articles [73, 74, 75], Peng donne dans un premier temps la définition de l'intégrale de G-Itô par les sommes de Riemann-Stieltjes pour le processus simple de la forme :

$$\eta_t = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k, t_{k+1})}(t), \tag{0.13}$$

où $\xi_k \in C_b(\Omega_{t_k})$. Dans la suite, l'espace $M_c^0([0,T])$ des processus de la forme (0.13) est complété pour la norme $(\frac{1}{T}\int_0^T \mathbb{E}[|\cdot|^2]dt)^{1/2}$ (cf. Peng [73, 74]) et pour la norme $(\frac{1}{T}\bar{\mathbb{E}}[\int_0^T |\cdot|^2 dt])^{1/2}$ (cf. Peng [75]), respectivement (les deux complétés sont notés par $\bar{M}_G^2([0,T])$ et $M_G^2([0,T])$, respectivement). Nous remarquons que les opérations de la G-espérance et de l'intégrale de Lebesgue ne sont pas permutables, et que la deuxième norme est un peu moins forte que la première. Après avoir obtenu une inégalité de G-Itô au lieu de l'isométrie d'Itô dans le cas classique, Peng démontre que l'intégrale de G-Itô

$$\mathcal{I}(\eta) = \int_0^T \eta dB_t$$

peut être vue comme une fonctionnelle continue et linéaire sur $M_c^0([0,T])$, et qu'elle peut être prolongée de façon unique aux complétés $\bar{M}_G^2([0,T])$ et $M_G^2([0,T])$ définis ci-dessus.

Par la suite, Li et Peng [52] donnent une définition de l'intégrale de G-Itô pour un processus de la forme (0.13), mais où les ξ_k sont remplacées par des variables aléatoires appartenant à $B_b(\Omega_{t_k})$. De façon similaire, ils montrent que la définition de l'intégrale de G-Itô peut être étendue au complété correspondant $M_*^2([0,T])$ pour la norme $(\frac{1}{T}\bar{\mathbb{E}}[\int_0^T |\cdot|^2 dt])^{1/2}$.

Nous remarquons qu'en général, une fonction indicatrice n'appartient pas à $L_G^1(\Omega_T)$, qui est le domaine le plus grand de $\mathbb{E}[\cdot]$. Une capacité de Choquet (cf. Choquet [12]) associée à la G-espérance $\mathbb{E}[\cdot]$ ne peut donc être définie que par rapport à l'espérance prolongée $\bar{\mathbb{E}}[\cdot]$ sur Ω :

$$\bar{C}(A) := \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{P}(A),\ A \in \mathcal{B}(\Omega).$$

Par rapport à cette capacité $\bar{C}(\cdot)$, une notion de « quasi-sûr » est établie dans l'article de Denis et al. [17] :

Définition 0.8 *Nous disons qu'un ensemble $A \in \mathcal{B}(\Omega)$ est polaire si et seulement si $\bar{C}(A) = 0$. De plus, nous disons qu'une propriété a lieu quasi-sûrement (q.s. en abrégé) si et seulement si elle a lieu en dehors d'un ensemble polaire.*

Grâce à la représentation de la G-espérance (0.12) et à la notion « quasi-sûr » introduite, la théorie des processus stochastiques en temps continu dans le cadre de la G-espérance s'est développée : en particulier la formule d'Itô, certaines inégalités stochastiques et les EDSs dirigées par un G-mouvement brownien (GEDSs en abrégé) peuvent être établies dans le sens « quasi-sûr ».

Une différence notable entre la G-espérance et celle de Wiener est que sous la G-espérance, la variation quadratique $\langle B\rangle$ du processus canonique B n'est plus le processus déterministe t. Par contre, les trajectoires de ce processus sont q.s. absolument continues par rapport au temps t, et donc l'intégrale par rapport à ce processus peut être construite au sens de Lebesgue-Stieltjes. Nous donnerons plus de détails dans la suite de cette thèse pour expliquer cette affirmation.

Après avoir défini les intégrales stochastiques dans le cadre de la G-espérance, il est naturel de considérer les GEDSs. Le premier travail sur les GEDSs est réalisé par Peng [73] en utilisant un argument de point fixe. Dans cet article, une équation de la forme suivante est donnée :

$$X_t = x + \int_0^t f(s,X_s)ds + \int_0^t h(s,X_s)d\langle B,B\rangle_s + \int_0^t g(s,X_s)dB_s,\ 0 \le t \le T, \tag{0.14}$$

sous l'hypothèse que les coefficients f, h et g sont uniformément lipschtiziens, comme dans le cas classique, et sous une hypothèse particulière sur leur régularité. Cette dernière s'écrit (dans le cas où le G-mouvement brownien et l'équation sont tous unidimensionnels) : pour tout $x \in \mathbb{R}$, les coefficients f, h et g vérifient :

$$f(\cdot,x),\ h(\cdot,x),\ g(\cdot,x) \in \bar{M}_G^2([0,T]). \tag{0.15}$$

Comme $\bar{\mathbb{E}}[\cdot]$ est une espérance supérieure, la norme $(\frac{1}{T}\bar{\mathbb{E}}[\int_0^T |\cdot|^2 dt])^{1/2}$ est plus forte que la norme classique sous chaque $\mathbb{P} \in \mathcal{P}_G$. L'espace d'intégration pour l'intégrale de G-Itô est donc plus petit que dans le cas classique. Dans le cadre classique, nous avons seulement besoin d'une hypothèse qui peut-être par exemple sur g : $E^{\mathbb{P}_0}[\int_0^T g(t,0)dt] < +\infty$, et grâce à l'hypothèse que g est uniformément lipschtizien, l'intégrabilité de g est obtenue pour chaque x, ce qui est suffisant pour montrer l'existence et l'unicité de la solution sur $[0,T]$. Mais dans le cadre de la G-espérance, même si à x fixé, toutes les trajectoires $g(\cdot,x)(\omega)$ sont continues, nous ne pouvons pas démontrer que $g(\cdot,x) \in \bar{M}_G^2([0,T])$ (même si cet espace est remplacé par $M_G^2([0,T])$ ou $M_*^2([0,T])$), et par conséquent nous ne sommes pas sûr que les intégrales de G-Itô soient bien définies lorsque nous construisons une suite de Picard pour l'itération, s'il nous manque ce type d'hypothèses (0.15).

Dans l'article de Peng [73], l'égalité de la GEDS (0.14) est au sens de la norme $\bar{M}_G^2([0,T])$, bien que ce ne soit pas intuitif. Après avoir étudié les propriétés des trajectoires de cette G-diffusion (0.14), Gao [27] propose de poser (0.14) dans le sens « quasi-sûr ». Nous remarquons que si les trajectoires de (0.14) sont assez régulières, les solutions dans les deux sens sont indistinguables. De plus, d'autres articles s'intéressent aux GEDSs, comme par exemple les articles de Guo et al. [30], Lin [54] et Lin et Bai [56]. Globalement, tous ces travaux sont réalisés sous une hypothèse commune qui assure que les coefficients sont à croissance linéaire.

Remarquons que dans ces articles sur les GEDSs, quelques outils puissants d'analyse stochastique dans le cadre de G-espérance sont développés. Par exemple, la formule de G-Itô est introduite dans l'article de Peng [74] et est généralisée par Gao [27], Li et Peng [52] et Zhang et al. [98]. Une inégalité de type Burkholder-Davis-Gundy est également prouvée dans ce nouveau cadre.

Enfin, il convient de noter que la théorie de la G-espérance a déjà trouvé des applications dans le domaine de la finance. Pour ces dernières, nous pouvons citer les articles d'Epstein et Ji [24, 25], et de Vorbrink [94]. Même si la recherche sur les applications de cette nouvelle théorie n'en est qu'à ses débuts, ces premiers travaux ont déjà ouvert de vastes perspectives.

0.2.2 Nouveaux résultats

Dans la première partie de cette thèse, nous présentons nos nouveaux résultats sur la théorie des GEDSs en trois chapitres : le chapitre 1 traite des GEDSs réfléchies unidimensionnelles ; le chapitre 2 traite des GEDSs dont les coefficients sont localement lipschitziens par des méthodes de localisation ; le chapitre 3 considère les GEDSs réfléchies dans le cas multidimensionnel.

GEDSs réfléchies unidimensionnelles

Le chapitre 1 traite de la GEDS réfléchie unidimensionnelle. Pour un certain $p > 2$, l'espace $\bar{M}_G^p([0,T])$ est défini d'une manière similaire que $\bar{M}_G^2([0,T])$. De plus, on désigne par $M_I([0,T])$ l'espace de processus continu est croissant. Nous considérons l'équation suivante :

$$X_t = x + \int_0^t f(s,X_s)ds + \int_0^t h(s,X_s)d\langle B\rangle_s + \int_0^t g(s,X_s)dB_s + K_t,\ 0 \le t \le T,\ q.s., \tag{0.16}$$

où la valeur initiale $x \in \mathbb{R}$, et pour un certain $p > 2$, les coefficients f, h et $g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ sont des fonctions qui vérifient pour tout $x \in \mathbb{R}$, $f(\cdot,x)$, $h(\cdot,x)$ et $g(\cdot,x) \in \bar{M}_G^p([0,T])$.

Soit S un processus d'obstacle. La solution de (0.16) est alors un couple (X,K) qui vérifie : pour le même p,

(i) $X \in \bar{M}_G^p([0,T])$ et $X_t \ge S_t$, $0 \le t \le T$, q.s. ;

(ii) $K \in M_I([0,T]) \cap \bar{M}_G^p([0,T])$ et $K_0 = 0$, q.s. ;

(iii) $\int_0^T (X_t - S_t)dK_t = 0$, q.s..

Dans un premier temps, nous établissons la notion d'intégrale stochastique par rapport à un processus continu et croissant dans le cadre de la G-espérance. Puis nous étudions les propriétés des intégrales de ce type, et donnons une extension de la formule de G-Itô pour un processus composé de la somme X d'un processus de G-Itô et d'un processus qui appartient à $M_I([0,T]) \cap \bar{M}_G^2([0,T])$:

$$X_t = X_s + \int_s^t f_u du + \int_s^t h_u d\langle B\rangle_u + \int_s^t g_u dB_u + K_t - K_s. \tag{0.17}$$

Théorème 0.9 *Soit $\Phi \in \mathcal{C}^2(\mathbb{R})$ une fonction réelle telle que Φ'' est à croissance polynomiale. Soient f, h et g des processus bornés dans $\bar{M}_G^2([0,T])$, et $K \in M_I([0,T]) \cap \bar{M}_G^2([0,T])$ tel que pour chaque $t \in [0,T]$,*

$$\lim_{s\to t} \bar{\mathbb{E}}[|K_t - K_s|^2] = 0; \tag{0.18}$$

et pour tout $p > 2$,

$$\bar{\mathbb{E}}[K_T^p] < +\infty. \tag{0.19}$$

Alors, pour un processus de G-Itô X de la forme (0.17), nous avons

$$\begin{aligned}\Phi(X_t) - \Phi(X_s) = &\int_s^t \frac{d\Phi}{dx}(X_u) f_u du + \int_s^t \frac{d\Phi}{dx}(X_u) h_u d\langle B\rangle_u \\ &+ \int_s^t \frac{d\Phi}{dx}(X_u) g_u dB_u + \int_s^t \frac{d\Phi}{dx}(X_u) dK_u \\ &+ \frac{1}{2}\int_s^t \frac{d^2\Phi}{dx^2}(X_u) g_u^2 d\langle B\rangle_u, \ q.s..\end{aligned}$$

La preuve de ce théorème est basée sur un théorème similaire pour un processus de G-Itô de l'article de Peng [75] : un point critique est donc de montrer la convergence suivante, dans $\bar{M}_G^2([0,T])$:

$$\left| \sum_{k=0}^{2^N-1} K_{t_k^{2^N}} \mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(\cdot) - K_\cdot \right| \to 0. \tag{0.20}$$

Dans la preuve de la formule d'Itô classique, ce type de convergence est facile à montrer en considérant la continuité des trajectoires de K et en utilisant le théorème de convergence dominée de Lebesgue. Cependant, en raison de l'absence d'un tel théorème dans le cadre de la G-espérance, nous supposons (0.18) afin d'assurer (0.20). Pour prouver ce théorème, nous commençons par approcher Φ par une suite $\{\Phi^N\}_{N\in\mathbb{N}}$ de fonctions dont les dérivées sont uniformément bornées et lipschitziennes. Afin de démontrer les trois convergences : $\Phi^N(X_t) \to \Phi(X_t)$ dans $L_G^2(\Omega_t)$; $\frac{d\Phi^N}{dx}(X) \to \frac{d\Phi}{dx}(X)$ et $\frac{d^2\Phi^N}{dx^2}(X) \to \frac{d^2\Phi}{dx^2}(X)$ dans $\bar{M}_G^2([0,T])$, nous avons besoin de l'hypothèse (0.19).

Dans un deuxième temps, nous examinons le G-mouvement brownien réfléchi. A l'aide de la résolution du problème de Skorokhod, nous établissons le théorème suivant :

Théorème 0.10 *Pour $p \geq 1$, il existe un unique couple (X, K) dans $\bar{M}_G^p([0,T]) \times (M_I([0,T]) \cap \bar{M}_G^p([0,T]))$, tel que*

$$X_t = B_t + K_t, \ 0 \leq t \leq T, \ \ q.s.,$$

où (a) X est positif ; (b) $K_0 = 0$; et (c) $\int_0^T X_t dK_t = 0$, q.s..

De la même manière, nous pouvons obtenir un théorème similaire pour un processus de G-Itô Y :

Théorème 0.11 *Pour $p > 2$, considérons un processus de G-Itô Y de la forme suivante, dont les coefficients appartiennent à $\bar{M}_G^p([0,T])$, et de valeur initiale $y \geq 0$:*

$$Y_t = y + \int_0^t f_s ds + \int_0^t h_s d\langle B\rangle_s + \int_0^t g_s dB_s, \ 0 \leq t \leq T.$$

Alors, il existe un unique couple (X, K) dans $\bar{M}_G^p([0,T]) \times (M_I([0,T]) \cap \bar{M}_G^p([0,T]))$, tel que

$$X_t = Y_t + K_t, \ 0 \leq t \leq T, \ \ q.s.,$$

où (a) X est positif ; (b) $K_0 = 0$; et (c) $\int_0^T X_t dK_t = 0$, q.s..

En supposant que les coefficients f, h et g sont uniformément lipchitziens par rapport à x et que l'obstacle S est un processus de G-Itô, nous démontrons que (0.16) admet une unique solution dans $\bar{M}_G^p([0,T])) \times (M_I([0,T]) \cap \bar{M}_G^p([0,T]))$. Nous présentons brièvement les étapes de la preuve : grâce au théorème 0.11, nous pouvons utiliser la résolution du problème de Skorokhod pour obtenir une représentation explicite de K par rapport à X qui est la solution de (0.16), puis nous établissons des estimations a priori, desquelles nous déduisons l'unicité de la solution. Par contre, pour l'existence de la solution, nous construisons une suite de Picard dont l'existence est assurée par le théorème 0.11, puis nous obtenons la convergence de cette suite par une méthode classique (cf. El Karoui et Chaleyat-Maurel [21]). En complément des résultats énoncés ci-dessus, un principe de comparaison est obtenu en utilisant la formule de G-Itô généralisée (cf. théorème 0.9).

Notons que notre méthode est utilisable si nous considérons (0.16) dans un cas symétrique, i.e. (0.16) avec un obstacle supérieur, car pour une GEDS réfléchie, le processus particulier lié au cadre de la G-espérance et le processus lié au problème réfléchi sont déjà séparés (contrairement à l'EDSR du seconde ordre réfléchie, cf. Matoussi et al. [61]) : démontrer le résultat d'existence et d'unicité de la solution en utilisant une méthode basée sur le théorème de point fixe ne nous cause donc pas de soucis.

Le chapitre 1 est organisé comme suit. Des résultats connus dans le cadre de la G-espérance sont rappelés dans la partie 1.2. La partie 1.3 est dédiée aux intégrales stochastiques par rapport à un processus croissant dans ce nouveau contexte et à une extension de la formule de G-Itô. Puis dans la partie 1.4 nous introduisons le résultat d'existence et d'unicité du G-mouvement brownien réfléchi. Enfin, la partie 1.5 présente nos résultats principaux sur la théorie des GEDSs réfléchies.

Méthodes de localisation pour les GEDSs

Le but du chapitre 2 est d'étudier une extension de la formule de G-Itô et une classe de GEDSs plus générales, dites GEDSs non-lipschitziennes, en utilisant des méthodes de localisation. Nous savons que dans le cadre classique, les intégrales d'Itô peuvent être définies pour les processus qui sont localement intégrables par rapport au mouvement brownien via une méthode de localisation. Dans le cadre de la G-espérance, le premier pas est posé dans l'article de Li et Peng [52] qui définissent l'intégrale de G-Itô pour de tels processus et prouvent une formule de G-Itô plus générale que celle de [75].

Dans le chapitre 2, nous commençons par quelques rappels sur l'idée de Li et Peng [52], puis nous introduisons la notion d'intégrale stochastique par rapport à un processus continu à variation finie dans le cadre de la G-espérance, qui est une généralisation de l'intégrale par rapport à un processus continu et croissant du chapitre 1. Parallèlement, une extension de la formule de G-Itô est donnée par la somme d'un processus de G-Itô et d'un processus continu à variation finie. La méthode de localisation nous permet de traiter le cas d'une fonction de classe $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)$, sans hypothèse supplémentaire sur ses dérivées. Dans le suite, nous adoptons souvent la convention de notation d'Einstein.

Théorème 0.12 *Soient $\Phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)$ et X un processus de la forme suivante :*

$$X_t^\nu = x_0^\nu + \int_0^t f_s^\nu ds + \int_0^t h_s^{\nu ij} d\langle B^i, B^j\rangle_s + \int_0^t g_s^{\nu j} dB_s^j + K_t^\nu,\ \nu = 1,\dots,n,$$

où f^ν et $h^{\nu ij}$ appartiennent à $M_w^1([0,T])$ (voir la définition 2.15), $g^{\nu j}$ appartient à $M_w^2([0,T])$, $\nu = 1,\dots,n$, i, $j = 1,\dots,d$, et K est un processus n-dimensionnel continu à variation finie vérifiant que pour une certaine constante positive α et tout $0 \le u_1 \le T$:

$$\lim_{u_2\to u_1} \bar{\mathbb{E}}[|V_0^{u_2}(K) - V_0^{u_1}(K)|^\alpha] = 0. \tag{0.21}$$

Alors, nous avons

$$\begin{aligned}\Phi(t,X_t) - \Phi(0,x_0) &= \int_0^t (\partial_t\Phi(s,X_s) + \partial_{x^\nu}\Phi(s,X_s) f_s^\nu) ds \\ &+ \int_0^t \left(\partial_{x^\nu}\Phi(s,X_s) h_s^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(s,X_s) g_s^{\mu i} g_s^{\nu j}\right) d\langle B^i, B^j\rangle_s \\ &+ \int_0^t \partial_{x^\nu}\Phi(s,X_s) g_s^{\nu j} dB_s^j + \int_0^t \partial_{x^\nu}\Phi(s,X_s) dK_s^\nu,\ q.s..\end{aligned}$$

Nous remarquons que l'hypothèse (0.21) sur K est affaiblie par rapport à (0.18) et (0.19). Pour prouver ce théorème, nous commencons avec un K borné, et grâce à la méthode de localisation via des temps d'arrêt, ce résultat préliminaire est généralisé aux K qui ne vérifient que (0.21).

Enfin, nous considérons les GEDSs dont les coefficients sont localement lipschitziens par rapport à x. Plus précisément, nous nous intéressons à la GEDS homogène en temps :

$$X_t = x + \int_0^t f(X_s)ds + \int_0^t h(X_s)d\langle B, B\rangle_s + \int_0^t g(X_s)dB_s,\ 0 \le t \le T,\ q.s., \tag{0.22}$$

dont les coefficients vérifient l'hypothèse suivante : les coefficients des matrices coefficients de (0.22) f^ν, h^ν_{ij} et $g^\nu_j : \mathbb{R}^n \to \mathbb{R}$, $\nu = 1, \ldots, n$, $i, j = 1, \ldots, d$, sont des fonctions déterministes, telles que, pour tout x, $x' \in \{x : |x| \le R\}$, il existe une constante positive C_R qui ne dépend que de R, telle que

$$|f(x) - f(x')| + ||h(x) - h(x')|| + ||g(x) - g(x')|| \le C_R|x - x'|.$$

Pour prouver l'existence de solutions, nous commençons par approcher la GEDS initiale (0.22) par une suite des solutions des GEDSs uniformément lipschitziennes suivantes :

$$X_t^N = x + \int_0^t f^N(X_s^N)ds + \int_0^t h^N(X_s^N)d\langle B, B\rangle_s + \int_0^t g^N(X_s^N)dB_s,\ 0 \le t \le T,\ q.s.,$$

où les coefficients f^N, h^N et g^N sont les fonctions tronquées, données génériquement par

$$\zeta^N(x) = \begin{cases} \zeta(x) & \text{, si } |x| \le N; \\ \zeta(Nx/|x|), & \text{si } |x| > N. \end{cases}$$

Puis nous définissons une suite de temps d'arrêts par

$$\tau_N := \inf\{t : |X_t^N| \ge N\} \wedge T.$$

En supposant que la GEDS initiale (0.22) possède une fonction de Lyapunov (ce qui nous permet de traiter une GEDS dont les coefficients sont à croissance non-linéaire) qui assure la non-explosion des solutions, nous démontrons que

$$\bar{C}\Big(\bigcup_{N=1}^{+\infty}\{\omega : \tau_N(\omega) = T\}\Big) = 1.$$

Cela nous permet de définir une solution de la GEDS initiale (0.22) sur un intervalle $[0, T]$, pour un T arbitraire, par la suite de solutions $\{X^N\}_{N\in\mathbb{N}}$. La preuve d'unicité de la solution est par contre triviale.

De plus, pour une GEDS non-homogène en temps vérifiant des hypothèses similaires, nous prouvons obtenir un résultat analogue.

Le chapitre 2 est organisé comme suit. Certains résultats connus dans le cadre de la G-espérance sont rappelés dans la partie 2.2. La partie 2.3 est dédiée aux intégrales de G-Itô pour une classe de processus « localement intégrables » par rapport au G-mouvement brownien. Puis dans la partie 2.4, nous définissons les intégrales stochastiques par rapport à un processus à variation finie dans ce cadre et donnons une extension de la formule de G-Itô. Nous présentons la théorie des GEDSs non-lipschitziennes dans la partie 2.5. Enfin, la partie 2.6 fournit quelques procédures complémentaires pour démontrer l'extension de la formule de G-Itô.

GEDSs réfléchies multidimensionelles

Dans le chapitre 3, nous étudions les GEDSs réfléchies multidimensionnelles sur l'espace $M^p_*([0, T]; \mathbb{R}^n)$ par une méthode de pénalisation introduite dans l'article de Menaldi [63]. Nous considérons la GEDS réfléchie n-dimensionnelle avec une contrainte ouverte et convexe $\mathcal{O}$ suivante :

$$X_t = x + \int_0^t f(s, X_s)ds + \int_0^t h(s, X_s)d\langle B, B\rangle_s + \int_0^t g(s, X_s)dB_s - K_t,\ 0 \le t \le T,\ q.s.. \tag{0.23}$$

Nous dirons qu'un couple de processus (X, K) à valeurs dans $\mathbb{R}^n \times \mathbb{R}^n$ résout cette GEDS réfléchie si

(i) X et K appartiennent à $M_*^2([0,T];\mathbb{R}^n)$. Pour tous les ω à l'extérieur d'un ensemble polaire A, les trajectoires $X_\cdot(\omega)$ et $K_\cdot(\omega)$ sont continues sur l'intervalle $[0,T]$;

(ii) Pour chaque $\omega \in A^c$, $X_\cdot(\omega)$ prend ses valeurs dans $\bar{\mathcal{O}}$, $K_\cdot(\omega)$ est à variation finie sur l'intervalle $[0,T]$ et $K_0(\omega) = 0$;

(iii) Pour tout Z qui est un processus tel que pour chaque $\omega \in A^c$, $Z_\cdot(\omega)$ prend ses valeurs dans $\bar{\mathcal{O}}$ et est continu, alors pour tout $t \in [0,T]$,

$$\int_0^t (X_t(\omega) - Z_t(\omega))dK_t(\omega) \geq 0, \text{ pour tout } \omega \in A^c.$$

L'idée pour prouver l'existence consiste à approcher la solution de (0.23) par la suite de solutions des GSDEs uniformément lipschitziennes pénalisée :

$$X_t^\varepsilon = x + \int_0^t f(s, X_s^\varepsilon)ds + \int_0^t g(s, X_s^\varepsilon)dB_s - \frac{1}{\varepsilon}\int_0^t \beta(X_s^\varepsilon)dt, \ 0 \leq t \leq T, \ q.s.,$$

où on désigne par $2\beta(x)$ le gradient du carré de la distance à $\mathcal{O}$. En supposant que les coefficients f, g et h sont lipschitziens et bornés, nous établissons les convergences uniformes en norme suivantes : pour $p \geq 1$,

$$\bar{\mathbb{E}}[\sup_{0\leq t\leq T} |X_t^\varepsilon - X_t^{\varepsilon'}|^p] \to 0, \text{ quand } \varepsilon, \ \varepsilon' \to 0;$$

$$\bar{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\frac{1}{\varepsilon}\int_0^t \beta(X_s^\varepsilon)ds - \frac{1}{\varepsilon'}\int_0^t \beta(X_s^{\varepsilon'})ds\right|^p\right] \to 0, \text{ quand } \varepsilon, \ \varepsilon' \to 0.$$

Alors, l'existence d'une solution de (0.23) est un corollaire direct. D'autre part, l'unicité de la solution est facile à montrer en utilisant la formule d'Itô sous chaque $\mathbb{P} \in \mathcal{P}_G$.

Le chapitre 3 est organisé en deux parties : la première présente la formulation des GEDSs réfléchies multidimensionnelles, tandis que la deuxième fournit les résultats de convergence.

0.3 Une étude des EDSRs du second ordre

0.3.1 Rappels sur la théorie des EDSRs du second ordre

En plus de la finance, la théorie des EDSRs trouve des applications dans le domaine des EDPs. Les premiers résultats d'une représentation probabiliste via la technique des EDSRs pour les solutions de viscosité des EDPs paraboliques semi-linéaires ont été obtenus dans les articles de Peng [70] et Pardoux et Peng [69], dans lesquels les auteurs considèrent une classe d'EDSRs markoviennes. Pour ces équations, les aléatoires du générateur et de la valeur terminale sont données par une diffusion. Plus précisément, on considère un couple (Y, Z) solution de l'EDSR suivante :

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s^{\mathbb{P}_0}, \ 0 \leq t \leq T, \ \mathbb{P}_0 - p.s., \tag{0.24}$$

où X est la solution d'une EDS :

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s^{\mathbb{P}_0}, \ 0 \leq t \leq T, \ \mathbb{P}_0 - p.s., \tag{0.25}$$

et f, g sont des fonctions déterministes continues. Le système, composé d'une EDS et d'une EDSR, est ce qu'on appelle un système d'équations différentielles stochastiques progressives-rétrogrades (EDSPRs en abrégé).

Nous considérons ensuite l'EDP suivante sur $[0,T) \times \mathbb{R}^d$:

$$\begin{cases} (\partial_t + \mathcal{L})u(t,x) + f(t,x,u(t,x), \nabla u(t,x)^{\mathbf{Tr}}\sigma(t,x)) = 0; \\ u(T,x) = g(x), \end{cases} \tag{0.26}$$

où $\mathcal{L}$ est le générateur infinitésimal associé à la diffusion (0.25), donné par

$$\mathcal{L}u(t,x) := \frac{1}{2}\mathrm{tr}(\sigma^{\mathbf{Tr}}(t,x)\sigma(t,x)D^2u(t,x)) + b(t,x)^{\mathbf{Tr}}\nabla u(t,x).$$

Lorsque f et les coefficients dans la définition de l'opérateur $\mathcal{L}$ sont assez réguliers, l'EDP (0.26) admet une solution classique v. En utilisant simplement la formule d'Itô, nous pouvons voir que $(v(t,X_t), \nabla v(t,X_t)^{\mathbf{Tr}}$ $\sigma(t,X_t))$ résout l'EDSR (0.24). En particulier, nous avons $v(0,x) = Y_0$, ce qui généralise la formule de Feynman-Kac qui donne une interprétation probabiliste à la solution d'une EDP.

Suite à ces premiers travaux, beaucoup d'articles se sont intéressés à l'interprétation probabiliste de solutions d'EDPs principalement dans les deux sens suivants : exprimer la correspondance entre une certaine classe d'EDPs et une certaine classe d'EDSRs, et obtenir des propriétés des EDPs (cf. Briand et Hu [8], Delbaen et al. [16], El Karoui et al. [22], Hu et Qian [38], Kobylanski [46] et Ma et al. [58]) ; proposer une méthode probabiliste de simulation numérique des solutions d'EDPs (cf. Bouchard et Touzi [5] et Zhang [97]).

Nous remarquons que cette interprétation probabiliste par rapport à un système d'EDSPR classique est seulement disponible pour les EDPs quasi-linéaires (où l'EDSPR est couplée, sinon, pour les EDPs semi-linéaires), i.e. les EDPs qui ont une dépendance linéaire par rapport aux dérivées du second ordre de la fonction inconnue. C'est pour cette raison qu'en utilisant la formule d'Itô, la Hessienne n'apparaît que dans le terme contenant la variation quadratique de X, et ce terme induit une dépendance linéaire de la Hessienne dans (0.26).

Pour ouvrir une voie vers l'interprétation probabiliste pour une plus grande classe d'EDPs, dites EDPs complètement non-linéaires, Cheredito et al. [11] eurent l'idée de construire un système d'EDSPRs dont le générateur de la partie rétrograde dépend de la variation quadratique de Z :

Définition 0.13 *Soient $(s,x) \in [0,T) \times \mathbb{R}^d$ et $(Y_t, Z_t, \Gamma_t, A_t)_{s \leq t \leq T}$ un quadruplet de processus $\mathcal{F}^s$-progre -ssivement mesurables (où $\{\mathcal{F}^s\}_{s \leq t \leq T}$ est la filtration naturelle complétée générée par mouvement brownien translaté $W_t^s := W_t - W_s$, $s \leq t \leq T$), qui prend ses valeurs dans $\mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \times \mathbb{R}^d$. Nous dirons que (Y, Z, Γ, A) est une solution d'une EDSR du second ordre (2EDSR en abrégé) correspondante aux paramètres $(X^{s,x}, h, g)$, si*

$$\begin{aligned} dY_t &= -h(t, X_t^{s,x}, Y_t, Z_t, \Gamma_t)dt + Z_t \circ dX_t^{s,x},\ s \leq t \leq T,\ \mathbb{P}_0 - p.s.; \\ dZ_t &= A_t dt + \Gamma_t dX_t^{s,x}; \\ Y_T &= g(X_T^{s,x}), \end{aligned} \tag{0.27}$$

où l'application $h : [0,T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ est continue, $Z \circ dX^{s,x}$ désigne l'intégrale de Stratonovich et $X^{s,x}$ est la solution de l'EDS suivante :

$$X_t = x + \int_s^t b(X_u)du + \int_s^t \sigma(X_u)dW_u^{\mathbb{P}_0},\ s \leq t \leq T,\ \mathbb{P}_0 - p.s.. \tag{0.28}$$

Nous définissons une EDP associée à la 2EDSR (0.27), qui permet d'avoir une dépendance complètement non-linéaire par rapport à la Hessienne de la fonction inconnue :

$$\begin{cases} \partial_t u(t,x) - h(t,x,u(t,x),\nabla u(t,x), D^2u(t,x)) = 0; \\ u(T,x) = g(x). \end{cases} \tag{0.29}$$

Soit $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ une fonction assez régulière qui résout l'EDP (0.29) : en utilisant encore la formule d'Itô, nous pouvons voir facilement que le quadruplet (Y, Z, Γ, A) défini par

$$Y_t := v(t, X_t^{s,x}),\ Z_t := \nabla v(t, X_t^{s,x}),\ \Gamma_t := D^2 v(t, X_t^{s,x}) \text{ et } A_t := \mathcal{L}\nabla v(t, X_t^{s,x}),\ s \leq t \leq T,\ \mathbb{P}_0 - p.s.,$$

résout la 2EDSR (0.27). Mais il ne s'agit que d'un exemple particulier. Cheridito et al. [11] considèrent un cas plus général, i.e. une interprétation probabiliste pour les solutions de viscosité des EDPs du type de (0.29) lorsque le générateur h est uniformément lipschitzien en y, à croissance polynomiale en x, z et γ, et décroissant en γ. Plus précisément, après avoir défini un ensemble admissible $\mathcal{A}^{s,x} := \cup_{m=1}^{+\infty} \mathcal{A}_m^{s,x}$ pour Z, où en fixant p_1 et $p_2 > 0$:

$$\mathcal{A}_m^{s,x} := \Big\{ Z : Z_t = z + \int_s^t A_u ds + \int_s^t \Gamma_u dX_u^{s,x},\ s \le t \le T,\ z \in \mathbb{R}^d, \\ \max\{|Z_t|, |A_t|, |\Gamma_t|\} \le m(1 + |X_t^{s,x}|^{p_1}); \\ |\Gamma_u - \Gamma_t| \le m(1 + |X_u^{s,x}|^{p_2} + |X_t^{s,x}|^{p_2})(|u-t| + |X_u^{s,x} - X_t^{s,x}|),\ s \le u \le t \le T \Big\},$$

les auteurs font remarquer que si une solution (Y, Z, Γ, A) de la 2EDSR (0.27) est déjà trouvée et est telle que le processus Z se situe dans $\mathcal{A}^{s,x}$, alors elle est la solution unique parmi tous les quadruplets dont Z appartient à cette contrainte $\mathcal{A}^{s,x}$. De plus, une solution de viscosité de l'EDP (0.29) peut être construite dans un certain sens à partir de cette solution (Y, Z, Γ, A).

Les questions suivantes se posent alors naturellement : l'unicité a-t-elle toujours lieu lorsque la valeur terminale n'est plus markovienne ou lorsque $\mathcal{A}^{s,x}$ est étendue à une classe moins technique. Malheureusement, Cheridito et al. [11] ne nous fournit pas de réponse. De plus, Soner et al. [87] nous donnent un contre-exemple lorsque la valeur terminale est une variable aléatoire appartenant à $L^2(\mathbb{P}_0)$. De plus, concernant l'existence de solutions pour les 2EDSRs de la forme (0.27), il n'existe à ce jour qu'un résultat dans un cas trivial dans l'article de Soner et al. [87], où $f(t, y, z, \gamma) = \frac{c}{2}$ pour une constante $c \neq 1$.

Il est clair que l'absence d'un résultat d'existence et d'unicité des solutions des équations de la forme (0.27) sous des hypothèses suffisamment faible diminue l'applicabilité de ces équations : nous pouvons donc nous demander s'il y a une nouvelle formulation équivalente à (0.27) qui remplit la même fonction.

Pour compléter les remarques précédentes, il est intéressant de noter que, en plus d'un schéma numérique probabiliste fournit dans l'article de Cheridito et al. [11], Fahim et al. [26] proposent un nouveau schéma, en combinant la méthode de Monte Carlo et des différences finies, pour des EDPs complètement non-linéaires du type (0.29), sans avoir rappelé la formulation de 2EDSRs. A partir de ce travail, Guyon et Labordère [31] effectuent une simulation du pricing dans un cadre à volatilité incertaine.

Nous rappelons l'idée de Cheridito et al. [11] : il s'agit d'ajouter une équation permettant d'avoir un contrôle sur la variation quadratique du processus Z et d'ajouter une variable libre dans le générateur h qui représente le coefficient de ce contrôle. Nous pouvons voir que dans l'EDSR de (0.27), ce contrôle agit sur le terme

$$(Z \circ X)_t := \int_t^T Z_s \circ dX_s,$$

qui peut être vu comme une fonctionnelle de Z et X. Il est naturel de se demander si nous pouvons atteindre les mêmes objectifs en contrôlant l'intégrateur X au lieu de Z et générer une formulation duale de ce problème.

En supposant que h dans (0.29) est convexe et décroissante en γ, d'après Rockafeller [81] nous avons la représentation suivante :

$$h(t, x, y, z, \gamma) = \sup_{a \in \mathbb{S}_d^{>0}} \Big\{ \frac{1}{2}\mathrm{tr}(a\gamma) - F(t, x, y, z, a) \Big\}, \tag{0.30}$$

où F est la transformée de Fenchel-Legendre de h. Nous pouvons tout d'abord considérer, pour une fonction déterministe $\hat{a}$ en t prenant ses valeurs dans $\mathbb{S}_d^{>0}$ (l'espace de matrices symétriques réelles définies positives d'ordre d), une EDP semi-linéaire sur $[0, T) \times \mathbb{R}^d$:

$$\begin{cases} \partial_t u(t, x) + \frac{1}{2}\mathrm{tr}(\hat{a}_t D^2 u(t, x)) - F(t, x, u(t, x), \nabla u(t, x), \hat{a}_t) = 0; \\ u(T, x) = g(x). \end{cases}$$

Il est facile de voir que cette EDP est associée à un système d'EDSPR classique :

$$Y_t = g(X_T) - \int_t^T F(s, X_s, Y_s, Z_s, \hat{a}_s) ds - \int_t^T Z_s dX_s,\ 0 \le t \le T,\ \mathbb{P}_0 - p.s.; \\ X_t = \int_0^t \hat{a}_s^{1/2} dW_s^{\mathbb{P}_0},\ 0 \le t \le T,\ \mathbb{P}_0 - p.s.,$$

où $\hat{a}$ est en fait la variation quadratique de X.

Remarquons que sous la probabilité $\mathbb{P}_0$, si nous faisons varier $\hat{a}$, nous pouvons obtenir une classe d'EDSRs dont les intégrateurs stochastiques sont des martingales contrôlées par une classe de $\hat{a}$. Dans ce cas, une solution Y doit être un processus universel pour toutes les EDSRs appartenant à cette classe, tel que chaque EDSR a lieu $\mathbb{P}_0$-p.s..

Or, pour construire une telle classe d'EDSRs, une autre méthode est adoptée par Soner et al. [87]. Elle consiste à travailler avec le processus canonique B au lieu de X, et à considérer une EDSR d'une forme universelle sous une classe de probabilités mutuellement singulières au lieu d'une seule probabilité $\mathbb{P}_0$. Plus précisément, les auteurs de [87] définissent une classe $\mathcal{P}_H$ (similaire à $\mathcal{P}_G$ dans le cadre de la G-espérance) composée des mesures de probabilité $\mathbb{P}$ sous lesquelles B est une « vraie » martingale, qui peut être représentée par

$$B_t = \int_0^t \hat{a}_s^{1/2} dW_s^{\mathbb{P}}, \tag{0.31}$$

où $\hat{a}$ est la densité de la variation quadratique de B, $W^{\mathbb{P}}$ est un mouvement brownien sous $\mathbb{P}$ défini par

$$W_t^{\mathbb{P}} = \int_0^t \hat{a}_s^{-1/2} dB_s,$$

et $\hat{a}_t \in D_{F_t}$, $0 \le t \le T$, $\mathbb{P}$-p.s., où D_{F_t} est le domaine de F en a indépendant de (y,z). Nous introduisons alors la définition suivante :

Définition 0.14 *Nous disons qu'une propriété a lieu $\mathcal{P}_H$-quasi-sûrement (q.s. en abrégé), si et seulement si elle a lieu $\mathbb{P}$-p.s., pour toute $\mathbb{P} \in \mathcal{P}_H$.*

Cette notion de « quasi-sûr » est légèrement plus faible que celle donnée dans le cadre de la G-espérance. Dans ce nouveau sens de « quasi-sûr » associé à la classe $\mathcal{P}_H$, une 2EDSR d'une nouvelle forme est posée :

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s, \hat{a}_s) ds + \int_t^T Z_s dB_s + K_T - K_t, \ \mathcal{P}_H - q.s., \tag{0.32}$$

où $\hat{a}$ est un processus défini ponctuellement qui coïncide $\mathbb{P}$-p.s. avec la densité de la variation quadratique de B sous chaque $\mathbb{P} \in \mathcal{P}_H$, et K est un processus croissant qui vérifie une condition de minimalité : pour chaque $\mathbb{P} \in \mathcal{P}_H$,

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} E_t^{\mathbb{P}'}[K_T^{\mathbb{P}'}], \ 0 \le t \le T, \ \mathbb{P} - p.s., \tag{0.33}$$

où

$$\mathcal{P}_H(t^+, \mathbb{P}) := \{\mathbb{P}' \in \mathcal{P}_H : \mathbb{P}'|_{\mathcal{F}_t^+} = \mathbb{P}|_{\mathcal{F}_t^+}\}.$$

Lorsque le générateur F est uniformément lipschitzien en (y,z), Soner et al. [87] ont démontré l'existence et l'unicité d'une solution à (0.32) sous une hypothèse particulière d'intégrabilité uniforme (en $\mathbb{P}$) sur ξ et F, et sous certaines hypothèses qui seront précisées ultérieurement, pour assurer la mesurabilité d'une solution construite trajectoire par trajectoire. Dans la partie de leur preuve concernant l'unicité, un théorème de représentation est donné, qui montre que la solution Y de (0.32) est en fait un supremum (dans un certain sens à préciser dans la suite) des solutions $y^{\mathbb{P}}$ des EDSRs classiques avec les mêmes paramètres (T, F, ξ) : pour chaque $\mathbb{P} \in \mathcal{P}_H$,

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} y_t^{\mathbb{P}'}(T, \xi), \ \mathbb{P} - p.s.. \tag{0.34}$$

De ce fait, nous pouvons noter que K est alors un « correcteur » qui compense la différence entre cette solution supérieure Y et les $Y^{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}_H$, et qui s'assure que l'égalité (0.32) est toujours vérifiée. D'autre part, l'existence d'une solution de (0.32) est aussi prouvée avec l'aide des solutions d'EDSRs classiques, mais par une technique moins habituelle, qui s'appelle « distributions probabilistes conditionnelles régulières (d.p.c.r. en abrégé) », permettant d'effectuer une analyse stochastique trajectoire par trajectoire et d'empêcher que nous ne rencontrions le problème de prouver une convergence uniforme (en $\mathbb{P}$) sous une classe de probabilités non-dominée. En effet, ce problème surgit si, pour montrer l'existence, nous suivons une procédure globale ayant pour base d'itération de Picard.

Au niveau de l'interprétation probabiliste de solutions d'EDPs, une formule du type Feynman-Kac est prouvée dans l'article [87]. Elle donne une description du lien entre les EDPs complètement non-linéaires et les EDSRs lorsque la fonction h est convexe et décroissante en γ.

Suite à ce premier résultat pour les 2EDSRs de la forme (0.32), plusieurs auteurs ont travaillé sur ce sujet afin d'obtenir un résultat similaire sous quelques hypothèses affaiblies, dans un cadre avec contraintes ou dans le cadre d'une filtration non-continue.

Le premier article est fait par Possamaï [77], qui traite du cas où le générateur F est uniformément continu en toutes les variables, à croissance linéaire en y, et vérifie une condition de monotonie. En déduisant un théorème de représentation similaire à (0.34) de Soner et al. [87], l'unicité de la solution pour la 2EDSR (0.32) est prouvée (cela peut être fait sans supposer la continuité uniforme en y) ; cependant, pour procéder comme dans l'article de Lepeltier et San Martín [47] puis étendre la preuve de l'existence d'une solution, utiliser un théorème de convergence monotone pour une suite décroissante est inéluctable. Comme nous l'avons déjà mentionné précédemment, Denis et al. [17] ont fourni un tel théorème pour une classe de probabilités non-dominée. Afin de travailler avec ce théorème, les auteur diminuent légèrement la classe de probabilités pour s'assurer que $\mathcal{P}_H$ est faiblement compact et renforcent un peu la condition d'intégrabilité de ξ et de F. Même si le sens de « quasi-sûr » ici n'est pas exactement le même que celui du cadre de la G-espérance, cette différence peut être négligée en appliquant ce théorème de convergence monotone à une suite $\{X^n\}_{n\in\mathbb{N}}$ de variables aléatoires qui sont toutes continues par rapport à ω.

Motivés par la résolution du problème de pricing d'options américaines dans un marché à volatilité incertaine, Matoussi et al. [61] ont considéré une nouvelle classe de 2EDSRs, dites 2EDSRs réfléchies. Ces équations sont définies avec un obstacle càdlàg S inférieur, le processus K permet donc en outre de s'assurer que la dynamique Y reste toujours au-dessus de l'obstacle S. La condition minimale (0.33) doit alors être réécrite : pour chaque $\mathbb{P} \in \mathcal{P}_H$,

$$K_t^{\mathbb{P}} - k_t^{\mathbb{P}} = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} E_t^{\mathbb{P}'}[K_T^{\mathbb{P}'} - k_T^{\mathbb{P}}],\ 0 \le t \le T,\ \mathbb{P}-p.s..$$

où k est un processus croissant qui appartient au triplet $(y^{\mathbb{P}}, z^{\mathbb{P}}, k^{\mathbb{P}})$, qui est la solution de l'EDSR réfléchie classique sous la probabilité $\mathbb{P}$ avec les mêmes paramètres (T, F, ξ, S) (dont l'existence et l'unicité sont assurées par Lepeltier et Xu [49]) :

$$\begin{cases} y_t^{\mathbb{P}} = \xi + \displaystyle\int_t^T F(s, y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, \hat{a}_s)ds - \int_t^T z_s^{\mathbb{P}} dB_s + k_T^{\mathbb{P}} - k_t^{\mathbb{P}},\ 0 \le t \le T,\ \mathbb{P}-p.s.; \\ y_t^{\mathbb{P}} \ge S_t;\ \displaystyle\int_0^t (y_{s^-}^{\mathbb{P}} - S_{s^-})dk_s^{\mathbb{P}} = 0,\ 0 \le t \le T,\ \mathbb{P}-p.s.. \end{cases}$$

Sous les mêmes hypothèses que dans l'article de Soner et al. [87] sur F, ξ et $\mathcal{P}_H$, le résultat d'existence et d'unicité pour les solutions de 2EDSRs réfléchies est obtenu après avoir établi un théorème de représentation et en utilisant un argument via la technique des d.p.c.r..

Toutefois, il convient d'indiquer que cet article [61] ne considère que le cas où l'obstacle est posé endessous. Dans le cas symétrique, il faut qu'un processus décroissant lié au problème réfléchi et un processus croissant lié au cadre de 2EDSRs contrôlent la dynamique en même temps, ce qui induit une discussion très complexe avec un processus qui ne possède que la finitude de ses variations comme propriété de régularité. Ce problème est resté ouvert jusqu'à ce que Matoussi et al. [60] remarquent qu'en supposant que l'obstacle supérieure peut être décomposé sous une certaine forme, ce processus à variation finie peut-être bien décomposé en deux parties dont les variations sont à supports disjoints. Ainsi, un résultat d'existence et d'unicité pour les solutions des 2EDSRs avec barrières des deux côtés est obtenu dans cet article et est utilisé pour résoudre un jeu de Dynkin incertain, et également pour résoudre un problème de sub-hedging (et super-hedging) d'options d'Israeli dans un marché à volatilité incertaine.

De plus, Possamaï et Zhou [78] traitent des 2EDSRs dont les générateurs F sont à croissance quadratique en z, et ce résultat est utilisé par Matoussi et al. [62] pour résoudre les problèmes de maximisation robuste d'utilité du portefeuille dans un marché à volatilité incertaine. Ces deux articles sont la base de notre travail, nous allons donc les introduire en détail afin de motiver les nouveaux résultats de la section 0.3.3.

Enfin, en considérant les 2EDSRs dirigées par un processus avec des sauts, cela nous permet de mieux modéliser le marché financier via la technique de 2EDSRs. Cependant, n'ayant pas travaillé dans ce cadre au cour de cette thèse, nous renvoyons le lecteur aux articles de Kazi-Tani et al. [44, 45].

0.3.2 Une voie vers les GEDSRs

Comme nous l'avons déjà expliqué dans la partie précédente, la 2EDSR (0.32) est étudiée avec une classe de probabilités non-dominées dans le sens « quasi-sûr » donné par la définition 0.14. Remarquons que le cadre de la G-espérance (cf. Denis et al. [17] et Peng [73, 74, 75]) concerne aussi l'analyse stochastique dans un sens similaire. Dans le cadre de la G-espérance, la notion d'EDSRs dirigées par un G-mouvement brownien (GEDSRs en abrégé) est bien définie. Pour $n = 1$ et $d = 1$, nous expliquons dans le suite le lien entre les deux types d'équations.

Nous considérons une fonction g définie sur $[0, T] \times \mathbb{R} \times \mathbb{R}$ et $H(t, y, z, \gamma) := G(\gamma) - g(t, y, z)$, où le noyau de la chaleur G est donné par

$$G(\gamma) := \frac{1}{2} \sup_{a \in [\underline{a}, \overline{a}]} (a\gamma) = \frac{1}{2}(\overline{a}\gamma^+ - \underline{a}\gamma^-),\ 0 < \underline{a} \leq \overline{a} < +\infty,$$

et D_H désigne le domaine de H en γ pour (y, z) fixé. D'après Rockafeller [81], nous avons la forme de représentation duale à (0.30) suivante :

$$F(t, y, z, a) = \sup_{\gamma \in D_H} \left\{ \frac{1}{2}(a\gamma) - H(t, y, z, \gamma) \right\} = \sup_{\gamma \in \mathbb{R}} \left\{ \frac{1}{2}(a\gamma) - H(t, y, z, \gamma) \right\}.$$

où F est la transformée de Fenchel-Legendre de H. Il est facile de voir que $F(t, y, z, a) := g(t, y, z)$ et que $D_{F_t} = [\underline{a}, \overline{a}]$, $0 \leq t \leq T$. Définissons une sous-classe $\tilde{\mathcal{P}}_H$ de $\mathcal{P}_H$, qui est composée des probabilités sous lesquelles $\hat{a}$ est uniformément majorée par $\overline{a}$ et uniformément minorée par $\underline{a}$, et considérons une 2EDSR dont le générateur est indépendant de a :

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t,\ \tilde{\mathcal{P}}_H - q.s.. \tag{0.35}$$

Si nous changeons un peu le sens de « quasi-sûr », cette équation est en fait une GEDSR. Considérons un cas plus simple, i.e., $g \equiv 0$. Dans ce cas, le problème d'existence et d'unicité des solutions de l'équation ci-dessus, avec une valeur terminale ξ, est équivalent au problème de décomposition de la martingale $Y_t = (\mathbb{E}[\xi|\Omega_t])_{0 \leq t \leq T}$:

$$Y_t = \xi - \int_t^T Z_s dB_s + K_T - K_t,\ \tilde{\mathcal{P}}_H - q.s..$$

D'après Soner et al. [85] et Song [89], si ξ appartient à l'ensemble $L_G^p(\Omega_T)$, $p > 1$, nous pouvons décomposer $(\mathbb{E}[\xi|\Omega_t])_{0 \leq t \leq T}$ d'une façon unique pour obtenir une martingale symétrique (qui peut être représentée par $\int ZdB$ selon Xu et Zhang [95]) et une martingale décroissante $-K$. En particulier, Peng [74] signale que si $\xi = \varphi(B_T)$, où φ est lipschitzienne et bornée, la martingale décroissante $-K$ peut encore être décomposée en

$$-K_t = -\frac{1}{2}\left(G(\eta_s) ds - \int_0^t \eta_s d\langle B \rangle_s \right),$$

où η est un certain processus dans $M_G^1([0, T])$. Plus récemment, Peng et al. [76] ont trouvé un sous-espace fermé de $L_G^p(\Omega_T)$, $p > 1$, et ont fourni un théorème de décomposition pour $(\mathbb{E}[\xi|\Omega_t])_{0 \leq t \leq T}$, où ξ appartient à ce sous-espace. Ce résultat est obtenu à l'aide d'une norme appropriée pour η introduite par Song [89] et d'une certaine estimation donnée par Hu et Peng [37].

Pour un cas plus général, i.e., g uniformément lipschitzienne en (y, z), Hu et al. [35] ont prouvé qu'il existe un unique triplet (Y, Z, K) qui vérifient (0.35), où $-K$ est une G-martingale décroissante. Dans cet article, l'unicité est montrée par des estimations a priori, tandis que l'existence est obtenue par une technique de partition de l'unité et l'approximation de Galerkin. Remarquons que dans ce cas, l'integrabilité de la valeur terminale n'est pas conforme à celle de la solution.

Un théorème de représentation pour la G-espérance conditionnelle est prouvé dans l'article de Soner et al.

[85] : par ce résultat, nous pouvons dire que K dans (0.35) vérifie toujours la condition minimale (0.33).

Notons de plus qu'en considérant un système composé d'une GEDSR et d'une GEDS, Hu et al. [36] et Peng [75] donnent une formule du type Feynman-Kac généralisée pour les EDPs complètement non-linéaires.

0.3.3 Motivations et nouveaux résultats

Nous avons déjà introduit dans la partie 0.1.1 des applications des EDSRs classiques afin de résoudre un problème de maximisation d'utilité de portefeuille. Ce modèle est établi sous une probabilité fixée $\mathbb{P}_0$, et il n'est donc utile que pour l'investisseur qui connaît très bien cette probabilité en considérant les données historiques du marché financier. Mais en réalité, il y a beaucoup de raisons qui font que l'investisseur ne peut pas la déterminer. Une façon plus robuste est alors de re-modéliser ce problème d'investissement avec une classe de probabilités et d'établir un modèle sous chaque $\mathbb{P}$. Ainsi, la fonction de valeur (0.6) est réécrite sous la forme suivante :

$$V(x) := \sup_{\pi \in \mathcal{A}} \inf_{\mathbb{P} \in \mathcal{P}_H} [U(X_T^{x,\pi})].$$

Notons $V^\pi(x) := \inf_{\mathbb{P} \in \mathcal{P}_H} [U(X_T^{x,\pi})]$: pour chaque π, $V^\pi(x)$ représente l'utilité du pire des cas, ce que nous optimisons. Pour cette raison, nous dirons que $V(x)$ est une fonction de valeur de « maximisation robuste ». Dans ce cas, les propriétés de la classe de probabilités $\mathcal{P}$ sont essentielles pour la résolution du problème.

Initialement, ce problème est traité lorsqu'il est composé de modèles dominés, c'est-à-dire qu'une mesure finie $\mathbb{P}_0$ peut-être trouvée telle que toutes les probabilités $\mathbb{P}$ appartenant à $\mathcal{P}$ soient absolument continues par rapport à celle-ci. Nous supposons que sous chaque $\mathbb{P}$, le processus de prix est du type (0.7) en utilisant le changement de mesure : nous pouvons voir que l'incertitude n'apparaît que dans le drift de cette diffusion S. Quelques travaux se sont attachés à la résolution de ce problème en utilisant des techniques d'analyse convexe, parmi lesquels Bordigoni, Matoussi et Schweizer [4] et Gundel [29].

La situation devient beaucoup plus compliquée si nous considérons un problème composé de modèles non-dominés. Le problème d'investissement optimal de ce type n'a été traité que récemment par Denis et Kervarec [18], en travaillant avec une classe de probabilités non-dominée mais faiblement compacte et avec une fonction d'utilité bornée, ce qui permet d'obtenir une propriété généralisée de « min-max » :

$$V(x) = \inf_{\mathbb{P} \in \mathcal{P}} \sup_{X \in \chi(x)} E^{\mathbb{P}}[U(X_T^{x,\chi})] = \sup_{X \in \chi(x)} \inf_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}[U(X_T^{x,\chi})],$$

où χ est un ensemble de processus de richesse admissibles. De la relation ci-dessus, les auteurs de [18] ont déduit qu'il existe une probabilité la moins favorable dans la classe $\mathcal{P}$ et ce problème peut donc être résolu par la résolution d'un problème classique sous cette probabilité.

Inspirés de la résolution d'un problème classique via la technique d'EDSRs, Matoussi et al. [62] ont développé très récemment un résultat analogue via la technique de 2EDSRs pour le problème composé de modèles non-dominés, avec des fonctions d'utilité particulières (exponentielle, puissance et logarithme). Nous considérons une utilité exponentielle dans la suite.

Supposons également qu'il n'y a qu'une seule obligation et d actifs sur le marché financier. Le taux d'intérêt de cette obligation est zéro et les processus de prix des actifs suivent les EDSs suivantes : sous chaque $\mathbb{P}$,

$$dS_t^i = S_t^i(b_t^i dt + dB_t^i),\ 0 \leq t \leq T,\ i = 1, \ldots, d,\ \mathbb{P} - p.s.,$$

D'après (0.31), nous savons que $\hat{a}^{1/2}$ joue en fait le rôle de la volatilité de l'article de Hu et al. [34]. De ce fait, en considérant ce processus de prix sous plusieurs $\mathbb{P}$, nous pouvons modéliser un problème de maximisation avec une incertitude sur la volatilité. Supposons de plus que l'investisseur détient un actif contingent ξ autre que son portefeuille, qui est à échéance au temps T. Matoussi et al. [62] cherchent une meilleure stratégie dans un ensemble $\tilde{\mathcal{A}}$ qui est composé des processus π $\mathcal{F}$-progressivement mesurables, à valeurs dans une contrainte $\tilde{C}$ et qui sont les générateurs de martingale OMB associée à la classe $\mathcal{P}$ (une notion étendue du générateur de martingale OMB classique), où π_t^i désigne le montant investi dans l'actif i au temps t. Alors, la fonction de valeur s'écrit :

$$V(x) := \sup_{\pi \in \tilde{\mathcal{A}}} \inf_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}\left[-\exp\left(-c\left(x + \int_0^T \pi_t(dB_t + b_t dt) - \xi \right)\right)\right]. \tag{0.36}$$

En travaillant avec une classe $\mathcal{P} := \tilde{\mathcal{P}}_H$ qui est définie dans la section précédente, Matoussi et al. démontrent que, soit lorsque ξ et b sont suffisamment petites et $0 \in \tilde{C}$, soit lorsque la frontière de $\tilde{C}$ est un arc de Jordan de type $\mathcal{C}^2$, il existe une meilleure stratégie π^* dans $\tilde{\mathcal{A}}$, qui optimise (0.36), et que la fonction valeur peut être représentée par la solution d'une 2EDSR dont le générateur est de la forme suivante : pour chaque $(\omega, t, z, a) \in \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{S}_d^{>0}$,

$$F(\omega, t, z, a) := \frac{c}{2} \operatorname{dist}^2 \left(a^{1/2} z + \frac{1}{c} a^{-1/2} b_t(\omega), a^{1/2} \tilde{C} \right) - z^{\mathbf{Tr}} b_t(\omega) - \frac{1}{2c} |a^{-1/2} b_t(\omega)|^2. \tag{0.37}$$

Par rapport aux résultats de Denis et Kervarec, l'approche qui se trouve dans l'article [62] permet de résoudre explicitement le problème avec certains types de fonctions d'utilité, mais les hypothèses supplémentaires adoptées sur ξ et b ou sur la frontière de $\tilde{C}$ ne sont ni assez pratiques en réalité, ni faciles à vérifier.

La raison pour laquelle ils adoptent ces hypothèses supplémentaires mentionnées ci-dessus est d'assurer l'existence et l'unicité de la solution pour la 2EDSR dont le générateur est de la forme (0.37) à croissance quadratique en z. Lorsque ce travail a été réalisé, il n'y avait aucun résultat sur les 2EDSRs de ce type, excepté celui obtenu par Possamaï et Zhou [78], qui traite d'une classe de 2EDSRs dont les générateurs F vérifient la condition suivante : F est lipschitzien en y, continue en z et il existe un triplet $(\alpha, \beta, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, tel que pour tout $(\omega, t, y, z, a) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times D_{F_t}$,

$$|F(\omega, t, y, z, a)| \leq \alpha + \beta |y| + \frac{\gamma}{2} |a^{1/2} z|^2. \tag{0.38}$$

Nous remarquons que sous chaque $\mathbb{P}$, cette hypothèse est la même que dans l'article de Morlais [64] pour les EDSRs dirigées par une martingale continue. Dans cet article [78], les auteurs obtiennent le résultat d'existence et d'unicité de la solution en supposant en plus de (0.38) une des deux hypothèses suivantes :

Hypothèse 0.15 *La valeur teminale ξ et $F(\cdot, \cdot, 0, 0, \hat{a}.(\cdot))$ sont suffisamment petites pour la norme sup, et F est « localement lipschitzien »en z, i.e. : pour tout $(\omega, t, y, z, z', a) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times D_{F_t}$,*

$$|F(\omega, t, y, z, a) - F(\omega, t, y, z', a) - a^{1/2}(z - z')| \leq C(|a^{1/2} z| + |a^{1/2} z'|)|a^{1/2}(z - z')|.$$

Hypothèse 0.16 *Le générateur F est de classe $\mathcal{C}^1$ en y et de classe $\mathcal{C}^2$ en z, et il existe deux constantes r et θ telles que, pour tout $(\omega, t, y, z, z', a) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times D_{F_t}$,*

$$|\partial_y F(\omega, t, y, z, a)| \leq r; \ |\partial_z F(\omega, t, y, z, a)| \leq r + \theta |a^{1/2} z|; \ |\partial^2_{zz} F(\omega, t, y, z, a)| \leq \theta.$$

Nous pouvons alors voir que les restrictions sur ξ, b et la frontière de $\tilde{C}$ permettent de s'assurer que le générateur F défini par (0.37) dans l'article de Matoussi et. al. [62] vérifie l'hypothèse 0.15 ou 0.16, de telle sorte que la 2EDSR associée à la résolution du problème de maximisation robuste admet une unique solution.

Afin de résoudre le problème financier mentionné ci-dessus sous des hypothèses plus naturelles, nous allons, dans le chapitre 4 de cette thèse, étudier les 2EDSRs dont les générateurs sont à croissance quadratique en z, en remplaçant les hypothèses 0.15 ou 0.16 par une autre hypothèse moins technique, et résoudre ensuite ce problème financier, mais sans les hypothèses supplémentaires de Matoussi et al. [62].

Il faut noter que les hypothèses 0.15 et 0.16 ne sont nécessaires que pour montrer la relation suivante dans la preuve d'existence d'une solution de l'article [78] : pour une probabilité fixée $\mathbb{P} \in \mathcal{P}_H$, tout $t \in [0, T]$ et $\mathbb{P}$-p.s. $\omega \in \Omega$,

$$y_t^{\mathbb{P}}(1, \xi)(\omega) = y_t^{\mathbb{P}^{t,\omega}, t, \omega}(1, \xi), \tag{0.39}$$

où $y_t^{\mathbb{P}}(1, \xi)$ est une solution d'une EDSR classique sur l'espace original, tandis que $y_t^{\mathbb{P}^{t,\omega}, t, \omega}(1, \xi)$ en est une autre sur l'espace translaté. Si la solution de l'EDSR sur l'espace original peut être construite par l'itération de Picard, cette relation est bien montrée en remplaçant les deux côtés par leur représentation sous forme d'espérance conditionnelle.

En se rappelant la théorie des EDSRs classiques, nous savons que Tevzadze [93] fournit un résultat d'existence et d'unicité par le théorème de point fixe pour les EDSRs dont les générateurs quadratique vérifient une hypothèse analogue à l'hypothèse 0.15. En utilisant ce résultat, cette relation a donc aussi lieu dans ce cas. Si l'hypothèse 0.15 est remplacée par l'hypothèse 0.16, les EDSRs correspondantes des deux côtés de

(0.39) peuvent être retrouvées par sommation des morceaux obtenus à partir des EDSRs initiales, chacun des morceaux étant une EDSR dont la valeur terminale a été divisée par un $N \in \mathbb{N}$ suffisamment grand. Alors, dans ce cas, la relation (0.39) a encore lieu.

Après avoir obtenu cette relation, un résultat d'existence et d'unicité de solutions pour les 2EDSRs quadratiques sous l'une ou l'autre des hypothèses mentionnées ci-dessus est établi dans l'article de Possamaï et Zhou [78], par la procédure introduite par Soner et al. [87].

Dans le chapitre 4 de cette thèse, nous introduisons une hypothèse pour remplacer les hypothèses 0.15 et 0.16, qui est plus usuelle et similaire à (0.5) du cas classique :

Hypothèse 0.17 *F est localement lipschitzien en z, i.e. : pour chaque $(\omega, t, y, z, z', a) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times D_{F_t}$,*

$$|F(t, \omega, y, z, a) - F(t, \omega, y, z', a)| \leq C(1 + |a^{1/2}z| + |a^{1/2}z'|)|a^{1/2}(z - z')|.$$

Sous cette hypothèse, nous démontrons que la relation (0.39) a également lieu, en utilisant un changement de variable exponentiel et une construction (similaire à (0.4)) de suites des solutions d'EDSRs à coefficients lipschitziens, et en appliquant le théorème de convergence monotone (cf. théorème 0.1).

Nous surmontons ensuite une difficulté sur la stabilité des EDSRs quadratiques, ainsi, l'existence de solutions pour les 2EDSRs de ce type est prouvée sans autre difficulté essentielle. D'autre part, l'unicité de la solution est démontrée via la technique de martingale OMB associée à la classe $\mathcal{P}_H$.

Après avoir généralisé le résultat théorique, nous traitons le problème financier. Nous considérons toujours la fonction de valeur (0.36), mais étendons légèrement l'ensemble des stratégies admissibles par la définition suivante :

Définition 0.18 *Soit $\tilde{C}$ un sous-ensemble fermé de $\mathbb{R}^d$. L'ensemble de stratégies admissibles $\tilde{\mathcal{A}}$ est composé des processus $\pi = \{\pi_t\}_{0 \leq t \leq 1}$ $\mathcal{F}$-progressivement mesurables prenant leurs valeurs dans $\tilde{C}$, $\lambda \otimes \mathcal{P}_H$-q.s., tels que pour chaque $\mathbb{P} \in \tilde{\mathcal{P}}_H$, $\int_0^1 |\hat{a}_t^{1/2}\pi_t|^2 dt < +\infty$, $\mathbb{P}$-p.s. et $\{\exp(-cX_\tau^\pi)\}_{\tau \in \mathcal{T}_0^1}$ est une famille $\mathbb{P}$-uniformément intégrable.*

Enfin, nous établissons le théorème suivant qui fournit une résolution au problème (0.36) et une forme explicite de stratégie optimale π^* qui appartient à l'ensemble $\tilde{\mathcal{A}}$, défini ci-dessus. Dans ce théorème, nous supposons que ξ s'adapte à la condition usuelle dans le cadre de 2EDSRs quadratiques et que b est borné :

Théorème 0.19 *Soit $\xi \in \mathcal{L}_{\tilde{H}}^\infty$. La fonction de valeur du problème de maximisation d'utilité du portefeuille (0.36) est donnée par*

$$V(x) = -\exp(-c(x - Y_0)),$$

où Y_0 est définie comme la solution unique $(Y, Z) \in \tilde{\mathbb{D}}_H^\infty \times \tilde{\mathbb{H}}_H^2$ de la 2EDSR suivante :

$$Y_t = \xi + \int_t^T F(s, Z_s, \hat{a}_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \leq t \leq T,\ \tilde{\mathcal{P}}_H - q.s.,$$

et où pour chaque $(\omega, t, z, a) \in \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{S}_d^{>0}$,

$$F(\omega, t, z, a) := \frac{c}{2}\operatorname{dist}^2\left(a^{1/2}z + \frac{1}{c}a^{-1/2}b_t(\omega), a^{1/2}\tilde{C}\right) - z^{\mathbf{Tr}}b_t(\omega) - \frac{1}{2c}|a^{-1/2}b_t(\omega)|^2.$$

De plus, la stratégie la plus optimale est donnée par

$$\hat{a}_t^{1/2}\pi_t^* \in \Pi_{\hat{a}_t^{1/2}\tilde{C}}\left(\hat{a}_t^{1/2}Z_t + \frac{1}{c}\hat{a}_t^{-1/2}b_t\right),\ \lambda \otimes \tilde{\mathcal{P}}_H - q.s., \tag{0.40}$$

où $\Pi_A(r)$ désigne la collection des points de l'ensemble fermé A qui réalisent la distance minimale entre l'ensemble A et le point r.

Dans la preuve du théorème ci-dessus, nous montrons bien que la stratégie π^* est un générateur de martingale OMB associée à la classe $\mathcal{P}_H$. En utilisant le théorème de représentation pour les solutions d'une 2EDSR quadratique, nous pouvons comparer cette valeur V^{π^*} associée à π^*, donnée par (0.40), à la valeur optimale du problème de maximisation classique sous chaque $\mathbb{P} \in \tilde{\mathcal{P}}_H$, puis déterminer que V^{π^*} est optimale pour le problème de maximisation robuste.

En plus du résultat ci-dessus, un problème de maximisation robuste avec une fonction de puissance est aussi examiné. Grâce à la théorie des 2EDSRs quadratiques généralisée dans un premier temps, nous pouvons traiter le cas d'une fonction de puissance avec le coefficient γ est inférieur à 1, au lieu de $\gamma < 0$ considéré dans l'article de Matoussi et al. [62]. A la fin de cette thèse, nous discutons la possibilité d'affaiblir l'hypothèse quadratique (0.38) et l'hypothèse 0.17 afin d'obtenir des résultats théoriques et appliqués plus généraux.

Le chapitre 4 est organisé comme suit. Des résultats connus concernant les 2EDSRs sont rappelés dans la partie 4.2. La partie 4.3 est dédiée aux estimations a priori et au résultat d'unicité pour les solutions de 2EDSRs à croissance quadratique. Dans la partie 4.4, nous déduisons le résultat d'existence de solutions pour les 2EDSRs à croissance quadratique. Enfin, la partie 4.5 traite des problèmes de maximisation robuste de l'utilité du portefeuille via la technique de 2EDSRs.

Part I

Stochastic Differential Equations Driven by G-Brownian motion

Chapter 1

Scalar Valued Reflected GSDEs

Abstract: In this chapter, we introduce the idea of stochastic integrals with respect to an increasing process in the G-framework and extend G-Itô's formula. Moreover, we study the solvability of the scalar valued stochastic differential equations driven by G-Brownian motion with reflecting boundary conditions (RGSDEs).

Key words. G-Brownian motion; G-expectation; increasing processes; G-Itô's formula; G-stochastic differential equations; reflecting boundary conditions.

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1.1 Introduction

In the classical framework, Skorokhod [83, 84] first introduced diffusion processes with reflecting boundaries in the 1960s. Since then, reflected solutions to stochastic differential equations (SDEs) and Backward SDEs (BSDEs) have been investigated by many authors. For the one-dimensional case, El Karoui [20], El Karoui and Chaleyat-Maurel [21] and Yamada [96] studied reflected SDEs (RSDEs) on a half-line and El Karoui et al. [22] obtained the solvability of reflected BSDEs. For the multidimensional case, the existence of weak solutions to reflected SDEs on a smooth domain was proved by Stroock and Varadhan [90]. Subsequently, Tanaka [92] solved the similar problem on a convex domain by a direct approach based on the solution to the Skorokhod problem. Furthermore, Lions and Sznitman [57] extended these results to a non-convex domain. The corresponding results for reflected BSDEs can be found in Gegout-Petit and Pardoux [28], Ramasubramanian [79] and Hu and Tang [39] and others.

Motivated by uncertainty problems, risk measures and super-hedging in finance, Peng [73, 75] introduced a framework of time consistent nonlinear expectation $\mathbb{E}[\cdot]$, i.e., G-expectation, in which a new type of Brownian motion was constructed and the corresponding stochastic calculus was established. In order to solve the super-replication problem in an uncertainty volatility model, Denis and Martini [19] independently introduced a notion of upper expectation and the related capacity theory. Moreover, a stochastic integral of Itô's type under a class of non-dominated probability measures was formulated. Recently, Denis et al. [17] found there is a strong link that connects these two frameworks, that is, the G-expectation $\mathbb{E}[\cdot]$ can be represented by a concrete weakly compact family $\mathcal{P}_G$ of probability measures:

$$\mathbb{E}[X] = \sup_{\mathbb{P}\in\mathcal{P}_G} E^{\mathbb{P}}[X],\ X \in L^1_G(\Omega).$$

Then, a Choquet capacity $\bar{C}(\cdot)$ can be naturally introduced to the G-framework:

$$\bar{C}(A) := \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{P}(A),\ A \in \mathcal{B}(\Omega),$$

by which we can have the following definition to the concept of "quasi-surely", similar to the one in Denis and Martini [19]: A set $A \subset \Omega$ is polar if $\bar{C}(A) = 0$; and a property holds "quasi-surely" (q.s. for short) if it holds outside a polar set. In these two frameworks, a stochastic integral of Itô type is defined following a usual procedure, that is, giving a definition first for some simple integrands and then completing the spaces of integrands under the norm induced by the upper expectation related to $\mathcal{P}_G$. This norm is much stronger than that in the classical case and thus, the space of integrands is smaller than the classical one. In other words, some additional regularity assumption should be imposed on the integrands to ensure that the integrals are well defined. Using these notions of stochastic calculus in the G-framework, the existence and uniqueness results for some types of SDEs driven by G-Brownian motion (GSDEs) can be obtained (cf. Peng [75], Gao [27] and Lin and Bai [56]). For the reason stated above, the authors who studied GSDEs always assumed the following condition on the coefficients of the equations: for each $x \in \mathbb{R}$,

$$f_\cdot(x),\ g_\cdot(x) \in \bar{M}^2_G([0,T]).$$

At this price, all results in the works for GSDEs listed above hold in the "quasi-surely" (q.s.) sense, i.e., outside a polar set, and all the processes are immediately aggregated.

Closely related to the G-framework, Soner et al. [88, 86, 87] have established another type of "quasi-sure" stochastic analysis and also a complete theory for second order BSDEs (2BSDEs) under a uniform Lipschitz condition on the coefficients. In that framework, another notion of "quasi-surely" was issued, which means that a property holds $\mathbb{P}$-a.s., for each probability measure $\mathbb{P} \in \mathcal{P}_H$, which is a class of local martingale measure. Obviously, this definition of "quasi-surely" is weaker than the one made by G-capacity. In this weaker sense, we can consider the stochastic integral with respect to the canonical B under each probability measure $\mathbb{P} \in \mathcal{P}_H$, respectively and we only need that these integrands meet the requirement for formulating a stochastic integral with respect to a local martingale. Thus, this type of setting for 2BSDEs ensures that we can treat the case that the coefficients have less regularity but that all the properties can only hold $\mathbb{P}$-a.s., for each $\mathbb{P} \in \mathcal{P}_H$. Following the pioneering work of Soner et al. [87], Matoussi et al. [61] have studied the problem of reflected 2BSDEs with a lower obstacle.

The aim of this chapter is to study the solvability of stochastic differential equations driven by G-Brownian motion with reflecting boundary conditions (RGSDEs) in the sense of "quasi-surely" defined by Denis et al. [17]. The scalar valued RGSDE that we consider is defined as following:

$$\begin{cases} X_t = x + \int_0^t f_s(X_s)ds + \int_0^t h_s(X_s)d\langle B\rangle_s + \int_0^t g_s(X_s)dB_s + K_t,\ 0 \leq t \leq T,\ q.s.; \\ X_t \geq S_t,\ 0 \leq t \leq T,\ q.s.;\ \int_0^T (X_t - S_t)dK_t = 0,\ q.s., \end{cases} \tag{1.1}$$

where $\langle B\rangle$ is the quadratic variation process of G-Brownian motion B and K is an increasing process that pushes the solution X upwards to remain above the obstacle S in a minimal way. Similarly to how the uniqueness results for classical reflected SDEs have been proved, the corresponding ones for RGSDEs can also be deduced from a priori estimates. Moreover, a solution in $\bar{M}_G^p([0,T])$ to (1.1) can be constructed by fixed-point iteration. Because of the reason that we have already explained, we need in addition to some assumption on the coefficients f, h and g, which is similar to that in Peng [75], Gao [27] and Lin and Bai [56], a regularity assumption on S to ensure that K stays in the space $\bar{M}_G^p([0,T])$. To establish the comparison theorem, we need to develop an extension of G-Itô's formula to deal with such a process X, which involves both stochastic integrals and an increasing process. This extended G-Itô's formula can have its own interest and may be used in other situations.

This chapter is organized as follows: Section 1.2 introduces notation and results in the G-framework which are necessary for what follows. Section 1.3 introduces the stochastic calculus with respect to an increasing process in the G-framework. Section 1.4 studies reflected G-Brownian motion and Section 1.5 presents our main results.

1.2 G-Brownian motion, G-capacity and G-stochastic calculus

The main purpose of this section is to recall some preliminary results in the G-framework, which are necessary later in the text. The reader interested in a more detailed description of these notions is referred to Denis et al. [17], Gao [27] and Peng [75].

1.2.1 G-Brownian motion

Adapting to the approach in Peng [75], let Ω be the space of all $\mathbb{R}$-valued continuous paths with $\omega_0 = 0$ equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N}((\max_{0\leq t\leq N} |\omega_t^1 - \omega_t^2|) \wedge 1),$$

B the canonical process and $C_{l,Lip}(\mathbb{R}^n)$ the collection of all local Lipschitz functions on $\mathbb{R}^n$. For a fixed $T > 0$, the space of finite dimensional cylinder random variables is defined by

$$L_{ip}^0(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \in \mathbb{N}^+,\ 0 \leq t_1 \leq \ldots \leq t_n \leq T,\ \varphi \in C_{l,Lip}(\mathbb{R}^n)\},$$

on which $\mathbb{E}[\cdot]$ is a sublinear functional that satisfies: for all $X, Y \in L_{ip}^0(\Omega_T)$,

(1) **Monotonicity:** if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;

(2) **Sub-additivity:** $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$;

(3) **Positive homogeneity:** $\mathbb{E}[\lambda X] = \lambda\mathbb{E}[X]$, for all $\lambda \geq 0$;

(4) **Constant translatability:** $\mathbb{E}[X + c] = \mathbb{E}[X] + c$, for all $c \in \mathbb{R}$.

The triple $(\Omega, L_{ip}^0(\Omega_T), \mathbb{E})$ is called a sublinear expectation space.

Definition 1.1 *A scalar valued random variable $X \in L_{ip}^0(\Omega_T)$ is G-normal distributed with parameters $(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, i.e., $X \sim \mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, if for each $\varphi \in C_{l,Lip}(\mathbb{R})$, $u^\varphi(t,x) := \mathbb{E}[\varphi(x + \sqrt{t}X)]$ is a viscosity solution to the following PDE on $\mathbb{R}^+ \times \mathbb{R}$:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - G\Big(\dfrac{\partial^2 u}{\partial x^2}\Big) = 0; \\ u|_{t=0} = \varphi, \end{cases}$$

where

$$G(a) := \frac{1}{2}(a^+\overline{\sigma}^2 - a^-\underline{\sigma}^2),\ a \in \mathbb{R}.$$

Remark 1.2 *Without loss of generality, we always assume that $\overline{\sigma}^2 = 1$ in what follows.*

Definition 1.3 *We call a sublinear expectation $\mathbb{E} : L_{ip}^0(\Omega_T) \to \mathbb{R}$ a G-expectation if the canonical process B is a G-Brownian motion under $\mathbb{E}[\cdot]$, that is, for each $0 \le s \le t \le T$, the increment $B_t - B_s \sim \mathcal{N}(0, [(t-s)\underline{\sigma}^2, (t-s)])$ and for all $n \in \mathbb{N}^+$, $0 \le t_1 \le \ldots \le t_n \le T$ and $\varphi \in C_{b,Lip}(\mathbb{R}^n)$,*

$$\mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_{n-1}}, B_{t_n} - B_{t_{n-1}})] = \mathbb{E}[\psi(B_{t_1}, \ldots, B_{t_{n-1}})],$$

where $\psi(x_1, \ldots, x_{n-1}) := \mathbb{E}[\varphi(x_1, \ldots, x_{n-1}, \sqrt{t_n - t_{n-1}}B_1)]$.

For $p \ge 1$, we denote by $L_G^p(\Omega_T)$ the completion of $L_{ip}^0(\Omega_T)$ under the Banach norm $\mathbb{E}[|\cdot|^p]^{\frac{1}{p}}$.

1.2.2 G-capacity

Derived in Denis et al. [17], G-expectation $\mathbb{E}[\cdot]$ can be viewed as an upper expectation $\bar{\mathbb{E}}[\cdot]$ associated with a weakly compact family $\mathcal{P}_G$ of probability measures on $L_G^1(\Omega_T)$, i.e.,

$$\mathbb{E}[X] = \bar{\mathbb{E}}[X] := \sup_{\mathbb{P} \in \mathcal{P}_G} E^{\mathbb{P}}[X],\ X \in L_G^1(\Omega_T).$$

In this sense, the domain of G-expectation can be extended from $L_G^1(\Omega_T)$ to the space of all $\mathcal{B}(\Omega_T)$ measurable random variables $L^0(\Omega_T)$ by setting

$$\bar{\mathbb{E}}[X] := \sup_{\mathbb{P} \in \mathcal{P}_G} E^{\mathbb{P}}[X],\ X \in L^0(\Omega_T).$$

Naturally, we can define a corresponding regular Choquet capacity on Ω:

$$\bar{C}(A) := \sup_{\mathbb{P} \in \mathcal{P}_G} \mathbb{P}(A),\ A \in \mathcal{B}(\Omega),$$

with respect to which, we have the following notions:

Definition 1.4 *A set $A \in \mathcal{B}(\Omega)$ is called polar if $\bar{C}(A) = 0$. A property is said to hold quasi-surely (q.s.) if it holds outside a polar set.*

Definition 1.5 *A random variable X is said to be quasi-continuous (q.c.) if for any arbitrarily small $\varepsilon > 0$, there exists an open set $O \subset \Omega$ with $\bar{C}(O) < \varepsilon$ such that X is continuous in ω on O^c.*

Definition 1.6 *We say that a random variable X has a q.c. version if there exists a q.c. random variable Y such that $X = Y$, q.s..*

In the language of G-capacity, Denis et al. [17] proved that for each $p \ge 1$, the function space $L_G^p(\Omega_T)$ has a dual representation, which is much more explicit to verify:

Theorem 1.7

$$L_G^p(\Omega_T) = \{X \in L^0(\Omega_T) : X\ has\ a\ q.c.\ version,\ \lim_{N \to +\infty} \bar{\mathbb{E}}[|X|^p \mathbf{1}_{|X|>N}] = 0\}.$$

Unlike in the classical framework, the downwards monotone convergence theorem only holds true for a sequence of random variables from a subset of $L^0(\Omega_T)$ (cf. Theorem 31 in Denis et al. [17]).

Theorem 1.8 *Let $\{X^n\}_{n \in \mathbb{N}} \subset L_G^1(\Omega_T)$ be such that $X^n \downarrow X$, q.s., then $\bar{\mathbb{E}}[X^n] \downarrow \bar{\mathbb{E}}[X]$.*

Remark 1.9 *We note that dominated convergence theorem does not exist in the G-framework, even though we assume that $\{X^n\}_{n \in \mathbb{N}}$ is a sequence in $L_G^1(\Omega_T)$. The lack of this theorem is one of the main difficulties we shall overcome in the following sections.*

1.2.3 G-stochastic calculus

In Peng [75], generalized Itô integrals with respect to G-Brownian motion are established:

Definition 1.10 *A partition of $[0,T]$ is a finite ordered subset $\pi_{[0,T]}^N := \{t_0, t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \ldots < t_N = T$. We set*

$$\mu(\pi_{[0,T]}^N) := \max_{k=0,1,\ldots,N-1} |t_{k+1} - t_k|.$$

For each $p \geq 1$, define

$$M_G^{p,0}([0,T]) := \left\{ \eta_t = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})}(t) : \xi_k \in L_G^p(\Omega_{t_k}) \right\},$$

and denote by $\bar{M}_G^p([0,T])$ the completion of $M_G^{p,0}([0,T])$ under the norm

$$||\eta||_{\bar{M}_G^p([0,T])} := \left(\frac{1}{T} \int_0^T \bar{\mathbb{E}}[|\eta_t|^p] dt \right)^{\frac{1}{p}}.$$

Remark 1.11 *By Definition 1.10, if η is an element in $\bar{M}_G^p([0,T])$, then there exists a sequence of processes $\{\eta^n\}_{n\in\mathbb{N}}$ in $M_G^{p,0}([0,T])$, such that $\lim_{n\to+\infty} \int_0^T \bar{\mathbb{E}}[|\eta_t^n - \eta_t|^p] dt \to 0$. It is easily observed that for almost every $t \in [0,T]$, $\{\eta_t^n\}_{n\in\mathbb{N}} \subset L_G^p(\Omega_t)$ and $\bar{\mathbb{E}}[|\eta_t^n - \eta_t|^p] \to 0$, thus η_t is an element in $L_G^p(\Omega_t)$.*

Definition 1.12 *For each $\eta \in M_G^{2,0}([0,T])$, we define*

$$\mathcal{I}_{[0,T]}(\eta) = \int_0^T \eta_s dB_s := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}} - B_{t_k}).$$

The mapping $\mathcal{I}_{[0,T]} : M_G^{2,0}([0,T]) \to L_G^2(\Omega_T)$ is continuous and linear and thus, can be uniquely extended to $\mathcal{I}_{[0,T]} : \bar{M}_G^2([0,T]) \to L_G^2(\Omega_T)$. Then, for each $\eta \in \bar{M}_G^2([0,T])$, the stochastic integral with respect to G-Brownian motion B is defined by $\int_0^T \eta_s dB_s := \mathcal{I}_{[0,T]}(\eta)$.

Unlike the classical theory, the quadratic variation process of G-Brownian motion B is not always a deterministic process (unless $\overline{\sigma} = \underline{\sigma}$) and it can be formulated in $L_G^2(\Omega_t)$ by

$$\langle B \rangle_t := \lim_{\mu(\pi_{[0,t]}^N)\to 0} \sum_{k=0}^{N-1} (B_{t_{k+1}^n} - B_{t_k^n})^2 = B_t^2 - 2\int_0^t B_s dB_s.$$

Definition 1.13 *For each $\eta \in M_G^{1,0}([0,T])$, we define*

$$\mathcal{Q}_{[0,T]}(\eta) = \int_0^T \eta_s d\langle B \rangle_s := \sum_{k=0}^{N-1} \xi_k (\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k}).$$

The mapping $\mathcal{Q}_{[0,T]} : M_G^{1,0}([0,T]) \to L_G^1(\Omega_T)$ is continuous and linear and thus, can be uniquely extended to $\mathcal{Q}_{[0,T]} : \bar{M}_G^1([0,T]) \to L_G^1(\Omega_T)$. Then, for each $\eta \in \bar{M}_G^1([0,T])$, the stochastic integral with respect to the quadratic variation process $\langle B \rangle$ is defined by $\int_0^T \eta_s d\langle B \rangle_s := \mathcal{Q}_{[0,T]}(\eta)$.

In view of the dual formulation of G-expectation, as well as the properties of the quadratic variation process $\langle B \rangle$ in the G-framework, the following BDG type inequalities are obvious.

Lemma 1.14 *Let $p \geq 1$, $\eta \in \bar{M}_G^p([0,T])$ and $0 \leq s \leq t \leq T$. Then,*

$$\bar{\mathbb{E}}\left[\sup_{s\leq u\leq t} \left| \int_s^u \eta_r d\langle B \rangle_r \right|^p \right] \leq |t-s|^{p-1} \int_t^s \bar{\mathbb{E}}[|\eta_u|^p] du.$$

Lemma 1.15 *Let $p \geq 2$, $\eta \in \bar{M}_G^p([0,T])$ and $0 \leq s \leq t \leq T$. Then,*

$$\bar{\mathbb{E}}\left[\sup_{s\leq u\leq t} \left| \int_s^u \eta_r dB_r \right|^p \right] \leq C_p \bar{\mathbb{E}}\left[\left| \int_s^t |\eta_u|^2 du \right|^{\frac{p}{2}} \right] \leq C_p |t-s|^{\frac{p}{2}-1} \int_s^t \bar{\mathbb{E}}[|\eta_u|^p] du,$$

where C_p is a positive constant independent of η.

1.3 Stochastic calculus with respect to an increasing process

In this section, we define stochastic integrals with respect to an increasing process with continuous paths, and then we extend G-Itô's formula to the case where an increasing process appears in the dynamics. In the following, C and M denote two positive constants whose values may vary from line to line.

1.3.1 Stochastic integrals with respect to an increasing process

Definition 1.16 *We denote by $M_c([0,T])$ the collection of all q.s. continuous processes X whose paths $X_\cdot(\omega): t \mapsto X_t(\omega)$ are continuous in t on $[0,T]$ outside a polar set A.*

Remark 1.17 *For example, from the proofs of Theorem 2.1 and 2.2 in Gao [27], $(\int_0^t \eta_s dB_s)_{0\le t\le T}$ and $(\int_0^t \eta_s d\langle B\rangle_s)_{0\le t\le T}$ have continuous modifications in $M_c([0,T])$.*

Definition 1.18 *We denote by $M_I([0,T])$ the collection of q.s. increasing processes $K \in M_c([0,T])$ whose paths $K_\cdot(\omega): t \mapsto K_t(\omega)$ are increasing in t on $[0,T]$ outside a polar set A.*

Remark 1.19 *Obviously, an increasing process K in $M_I([0,T])$ has q.s. finite total variation on $[0,T]$ and thus, its quadratic variation is q.s. 0.*

Definition 1.20 *We define, for a fixed $X \in M_c([0,T])$, the stochastic integral with respect to a given $K \in M_I([0,T])$ by*

$$\Big(\int_0^T X_t dK_t\Big)(\omega) = \begin{cases} \displaystyle\int_0^T X_t(\omega) dK_t(\omega)\ ,\ \omega \in A^c; \\ \qquad\qquad 0\ ,\ \omega \in A, \end{cases} \tag{1.2}$$

where A is a polar set and on the complementary of which, $X_\cdot(\omega)$ is continuous and $K_\cdot(\omega)$ is continuous and increasing in t.

Remark 1.21 *Because for a fixed $\omega \in A^c$, the function $X_\cdot(\omega)$ is continuous and the function $K_\cdot(\omega)$ is of bounded variation on $[0,T]$, the Riemann-Stieltjes integral on the right-hand side always exists (cf. Hildebrandt [33]). Thus, (1.2) is well defined. Similar definitions can be made for those X whose paths are q.s. piecewisely continuous and without discontinuity of the second kind , i.e., for each $\omega \in A^c$, the function $X_\cdot(\omega)$ is discontinuous at a finite number of points and these discontinuous points are removable or of the first kind.*

Remark 1.22 *Given a sequence of refining partitions $\{\pi^N_{[0,T]}\}_{N\in\mathbb{N}}$ (i.e., $\pi^N_{[0,T]} \subset \pi^{N+1}_{[0,T]}$, for all $N \in \mathbb{N}$) such that $\mu(\pi^N_{[0,T]}) \to 0$, we set a sequence of binary functions:*

$$\mathcal{V}^N_{[0,T]}(X,K)(\omega) := \sum_{k=0}^{N-1} X_{u^N_k}(\omega)(K_{t^N_{k+1}}(\omega) - K_{t^N_k}(\omega)), \tag{1.3}$$

where $u^N_k \in [t^N_k, t^N_{k+1})$. For a fixed $\omega \in A^c$, by the Heine-Cantor theorem, $X_\cdot(\omega)$ and $K_\cdot(\omega)$ are uniformly continuous in t on $[0,T]$. Therefore, we can find an $M_\omega > 0$ such that $K_T(\omega) < M_\omega$, then, for any arbitrarily small $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $|t-s| < \delta$, $|X_t(\omega) - X_s(\omega|) < \varepsilon/M_\omega$. It is sufficient to choose an $N_0 \in \mathbb{N}$ such that $\mu(\pi^{N_0}_{[0,T]}) < \delta$, then, for all $N > N_0$,

$$\Big|\mathcal{V}^N_{[0,T]}(X,K)(\omega) - \Big(\int_0^T X_t dK_t\Big)(\omega)\Big| < \varepsilon,$$

from which we deduce

$$\mathcal{V}^N_{[0,T]}(X,K) \to \int_0^T X_t dK_t,\ q.s.,\ as\ N \to +\infty. \tag{1.4}$$

The construction of sequence (1.3) provides a q.s. approximation to the stochastic integral $\int_0^T X_t dK_t$. We note that the convergence (1.4) depends only on the sequence of refined partitions $(\pi^N_{[0,T]})_{N\in\mathbb{N}}$ but is independent of the selection of the points of division and the representatives $X_{u^N_k}$ on $[t^N_k, t^N_{k+1})$, $k = 0, 1, \ldots, N-1$, $N \in \mathbb{N}$.

The following propositions can be verified directly by Definition 1.20 and the Heine-Cantor theorem.

Proposition 1.23 *Let X, X^1, $X^2 \in M_c([0,T])$, K, K^1, $K^2 \in M_I([0,T])$ and $0 \le s \le r \le t \le T$, then we have*

(1) $\int_s^t X_u dK_u = \int_s^r X_u dK_u + \int_r^t X_u dK_u$, *q.s.;*

(2) $\int_s^t (\alpha X_u^1 + X_u^2) dK_u = \alpha \int_s^t X_u^1 dK_u + \int_s^t X_u^2 dK_u$, *q.s., where* $\alpha \in L^0(\Omega_s)$*;*

(3) $\int_s^t X_u d(K^1 \pm K^2)_u = \int_s^t X_u dK_u^1 \pm \int_s^t X_u dK_u^2$, *q.s..*

Remark 1.24 *By a classical argument, a q.s. continuous and bounded variation process can be viewed as the difference of two increasing processes $K_1 - K_2$, where K_1, $K_2 \in M_I([0,T])$. By Proposition 1.23 (3), the stochastic integral with respect to $K_1 - K_2$ can be defined in the same way as Definition 1.20.*

Proposition 1.25 *Let $X \in M_c([0,T])$ and $K \in M_I([0,T])$, then the integral $\int_0^{\cdot} X_s dK_s$ is q.s. continuous in t, i.e., $(\int_0^t X_s dK_s)_{0 \le t \le T} \in M_c([0,T])$.*

As shown above, (1.2) defines a random variable $\int_0^T X_t dK_t$ in $L^0(\Omega_T)$. A natural question arises: if we assume that for some appropriate p and q, $X \in \bar{M}_G^p([0,T])$ and $K \in \bar{M}_G^q([0,T])$, can this random variable $\int_0^T X_t dK_t$ be verified as an element in $L_G^1(\Omega_T)$ or not? In general, the answer is negative. This is because the integrability of X and K cannot ensure the quasi-continuity of $\int_0^T X_t dK_t$ (cf. Definition 1.5 and Theorem 1.7). More precisely, the pathwise convergence (1.4) is not necessarily uniform in ω outside a polar set A and it is hard to verify directly the convergence in the sense of $L_G^1(\Omega_T)$ due to the lack of the dominated convergence theorem in the G-framework. However, in some special cases, a proper sequence $\{\mathcal{V}_{[0,T]}^N(X,K)\}_{N\in\mathbb{N}}$ approximating to $\int_0^T X_t dK_t$ can be found and thus, the quasi-continuity is inherited during the approximation.

Proposition 1.26 *Let $K \in M_I([0,T]) \cap \bar{M}_G^2([0,T])$, $K_T \in L_G^2(\Omega_T)$ and $\phi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, then $\int_0^T \phi(K_t) dK_t$ is an element in $L_G^1(\Omega_T)$.*

Proof: Consider a sequence of refining partitions $\{\pi_{[0,T]}^N\}_{N\in\mathbb{N}}$ mentioned in Remark 1.22 and define the sequence of approximation: for each $N \in \mathbb{N}$,

$$\mathcal{V}_{[0,T]}^N(\phi(K),K)(\omega) = \sum_{k=0}^{N-1} \phi(K_{t_k^N})(\omega)(K_{t_{k+1}^N}(\omega) - K_{t_k^N}(\omega)).$$

From the explanation in Remark 1.11, we can always assume that at the points of division, $K_{t_k^N} \in L_G^2(\Omega_T)$, $k = 0, 1, \ldots, N-1$, $N \in \mathbb{N}$. As K is increasing, we have

$$\begin{aligned} \left| \mathcal{V}_{[0,T]}^N(\phi(K),K) - \int_0^T \phi(K_t) dK_t \right| &\le \left| \int_0^T \left(\sum_{k=0}^{N-1} |K_{t_{k+1}^N} - K_{t_k^N}| \mathbf{1}_{[t_k^N, t_{k+1}^N)}(t) \right) dK_t \right| \\ &\le \sum_{k=0}^{N-1} |K_{t_{k+1}^N} - K_{t_k^N}|^2 \downarrow 0, \ q.s., \text{ as } N \to +\infty. \end{aligned}$$

On the other hand, it is easy to verify by Theorem 1.7 that for all $N \in \mathbb{N}$, $\mathcal{V}_{[0,T]}^N(\phi(K),K)$ and $\sum_{k=0}^{N-1} |K_{t_{k+1}^N} - K_{t_k^N}|^2 \in L_G^1(\Omega_T)$. Then, by Theorem 1.8, we obtain

$$\bar{\mathbb{E}}\left[\left| \mathcal{V}_{[0,T]}^N(\phi(K),K) - \int_0^T \phi(K_t) dK_t \right| \right] \le \bar{\mathbb{E}}\left[\sum_{k=0}^{N-1} |K_{t_{k+1}^N} - K_{t_k^N}|^2 \right] \downarrow 0, \text{ as } N \to +\infty.$$

From the completeness of $L_G^1(\Omega_T)$ under $\bar{\mathbb{E}}[|\cdot|]$, we deduce the desired result. □

Remark 1.27 *To verify that for all $N \in \mathbb{N}$, $\mathcal{V}_{[0,T]}^N(\phi(K),K)$ and $\sum_{k=0}^{N-1} |K_{t_{k+1}^N} - K_{t_k^N}|^2 \in L_G^1(\Omega_T)$, we should assume here that $K_T \in L_G^2(\Omega_T)$.*

Proposition 1.28 *Let X be a q.s. continuous G-Itô process such that*

$$X_t = x + \int_0^t f_s ds + \int_0^t h_s d\langle B\rangle_s + \int_0^t g_s dB_s,\ 0 \le t \le T, \tag{1.5}$$

where f, h and g are elements in $\bar{M}_G^p([0,T])$, $p > 2$. Let $K \in M_I([0,T]) \cap \bar{M}_G^q([0,T])$ and $K_T \in L_G^q(\Omega_T)$, where $1/p + 1/q = 1$. Then, $\int_0^T X_t dK_t$ is an element in $L_G^1(\Omega_T)$.

Proof: Given a sequence of refining partitions $\{\pi_{[0,T]}^N\}_{N\in\mathbb{N}}$, we construct sequence (1.3). By the definitions of G-stochastic integrals and the BDG type inequalities, one can verify that for each $t \in [0,T]$, $X_t \in L_G^p(\Omega_t)$. Therefore, for all $N \in \mathbb{N}$, $\mathcal{V}_{[0,T]}^N(X,K) \in L_G^1(\Omega_T)$. Applying the BDG type inequalities, we have

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{s\le u\le t} |X_u - X_s|^p] \le C\bigg(|t-s|^{p-1}\bigg(\int_s^t (\bar{\mathbb{E}}[|f_u|^p] + \bar{\mathbb{E}}[|h_u|^p])du\bigg) \\ + |t-s|^{\frac{p}{2}-1}\int_s^t \bar{\mathbb{E}}[|g_u|^p]du\bigg).\end{aligned}$$

Thus,

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{k\in[0,N)\cap\mathbb{N}}\sup_{t_k\le t\le t_{k+1}} |X_t - X_{t_k^N}|^p] &\le \bar{\mathbb{E}}\Big[\sum_{k=0}^{N-1}\sup_{t_k^N\le t\le t_{k+1}^N} |X_t - X_{t_k^N}|^p\Big] \\ &\le C\sum_{k=0}^{N-1}\Big(\int_{t_k^N}^{t_{k+1}^N} (|t_{k+1}^N - t_k^N|^{p-1}(\bar{\mathbb{E}}[|f_t|^p] + \bar{\mathbb{E}}[|h_t|^p]) + |t_{k+1}^N - t_k^N|^{\frac{p}{2}-1}\bar{\mathbb{E}}[|g_t|^p])dt\Big) \\ &\le C\Big(\mu(\pi_{[0,T]}^N)^{p-1}\int_0^T (\bar{\mathbb{E}}[|f_t|^p] + \bar{\mathbb{E}}[|h_t|^p])dt + \mu(\pi_{[0,T]}^N)^{\frac{p}{2}-1}\int_0^T \bar{\mathbb{E}}[|g_t|^p]dt\Big).\end{aligned} \tag{1.6}$$

From the integrability of f, h and g, we have

$$\bar{\mathbb{E}}[\sup_{k\in[0,N)\cap\mathbb{N}}\sup_{t_k\le t\le t_{k+1}} |X_t - X_{t_k^N}|^p] \le CM(\mu(\pi_{[0,T]}^N)^{p-1} + \mu(\pi_{[0,T]}^N)^{\frac{p}{2}-1}).$$

For each $N \in \mathbb{N}$, we calculate

$$\begin{aligned}\left|\mathcal{V}_{[0,T]}^N(X,K) - \int_0^T X_t dK_t\right| &\le \int_0^T \left|\sum_{k=0}^{N-1} X_{t_k^N}\mathbf{1}_{[t_k^N,t_{k+1}^N)}(t) - X_t\right| dK_t \\ &\le \sup_{0\le t\le T}\left|\sum_{k=0}^{N-1} X_{t_k^N}\mathbf{1}_{[t_k^N,t_{k+1}^N)}(t) - X_t\right| K_T \\ &\le K_T \sup_{k\in[0,N)\cap\mathbb{N}}\sup_{t_k\le t<t_{k+1}} |X_t - X_{t_k^N}|.\end{aligned}$$

Consequently,

$$\begin{aligned}\bar{\mathbb{E}}[|\mathcal{V}_{[0,T]}^N(X,K) - \textstyle\int_0^T X_t dK_t|] &\le \bar{\mathbb{E}}[K_T \sup_{k\in[0,N)\cap\mathbb{N}}\sup_{t_k\le t<t_{k+1}} |X_t - X_{t_k^N}|] \\ &\le (\bar{\mathbb{E}}[\sup_{k\in[0,N)\cap\mathbb{N}}\sup_{t_k\le t<t_{k+1}} |X_t - X_{t_k^N}|^p])^{\frac{1}{p}}(\bar{\mathbb{E}}[K_T^q])^{\frac{1}{q}} \\ &\le CM(\mu(\pi_{[0,T]}^N)^{p-1} + \mu(\pi_{[0,T]}^N)^{\frac{p}{2}-1})^{\frac{1}{p}} \to 0,\ \text{as } N \to +\infty.\end{aligned}$$

The desired result follows. □

1.3.2 An extension of G-Itô's formula

For each $0 \le s \le t \le T$, consider a sum of a G-Itô process and an increasing process K:

$$X_t = X_s + \int_s^t f_u du + \int_s^t h_u d\langle B\rangle_u + \int_s^t g_u dB_u + K_t - K_s.$$

Lemma 1.29 *Let $\Phi \in \mathcal{C}^2(\mathbb{R})$ be a real function with bounded and Lipschitz derivatives. Let f, h and g be bounded processes in $\bar{M}_G^2([0,T])$ and $K \in M_I([0,T]) \cap \bar{M}_G^2([0,T])$ satisfy for each $t \in [0,T]$,*

$$\lim_{s\to t} \bar{\mathbb{E}}[|K_t - K_s|^2] = 0. \tag{1.7}$$

Then,

$$\begin{aligned}\Phi(X_t) - \Phi(X_s) = & \int_s^t \frac{d\Phi}{dx}(X_u) f_u du + \int_s^t \frac{d\Phi}{dx}(X_u) h_u d\langle B\rangle_u \\ & + \int_s^t \frac{d\Phi}{dx}(X_u) g_u dB_u + \int_s^t \frac{d\Phi}{dx}(X_u) dK_u \\ & + \frac{1}{2}\int_s^t \frac{d^2\Phi}{dx^2}(X_u) g_u^2 d\langle B\rangle_u,\ q.s..\end{aligned} \tag{1.8}$$

The proof of this lemma is based on previous results in Peng [75] (cf. Lemma 6.1 and Proposition 6.3 in Chapter III). To avoid redundancy, we first prove a reduced lemma when $f = h = g \equiv 0$ to show how the increasing process K plays a role in this dynamic and then give a sketch to indicate some key points to combine the simple lemma with the previous results in Peng [75].

Lemma 1.30 *Let $\Phi \in \mathcal{C}^2(\mathbb{R})$ be a real function with bounded and Lipschitz derivatives and $K \in M_I([0,T]) \cap \bar{M}_G^2([0,T])$. Then,*

$$\Phi(K_t) - \Phi(K_s) = \int_s^t \frac{d\Phi}{dx}(K_u) dK_u,\ q.s..$$

Proof: Consider a sequence of refining partitions $\{\pi_{[s,t]}^N\}_{N\in\mathbb{N}}$. For each $N \in \mathbb{N}$, from the second order Taylor expansion, we have

$$\begin{aligned}\Phi(K_t) - \Phi(K_s) &= \sum_{k=0}^{N-1} (\Phi(K_{t_{k+1}^N}) - \Phi(K_{t_k^N})) \\ &= \sum_{k=0}^{N-1} \frac{d\Phi}{dx}(K_{t_k^N})(K_{t_{k+1}^N} - K_{t_k^N}) + \frac{1}{2}\sum_{k=0}^{N-1} \frac{d^2\Phi}{dx^2}(\xi_k^N)(K_{t_{k+1}^N} - K_{t_k^N})^2,\end{aligned}$$

where ξ_k^N satisfies $K_{t_k^N} \le \xi_k^N \le K_{t_{k+1}^N}$, q.s.. For the first part, similar to that in Remark 1.22, we obtain

$$\lim_{N\to+\infty}\left| \sum_{k=0}^{N-1} \frac{d\Phi}{dx}(K_{t_k^N})(K_{t_{k+1}^N} - K_{t_k^N}) - \int_s^t \frac{d\Phi}{dx}(K_u) dK_u \right| = 0,\ q.s..$$

For the second part, because $\frac{d^2\Phi}{dx^2}$ is bounded and the quadratic variation of K on $[0,T]$ is q.s. 0, then,

$$\frac{1}{2}\sum_{k=0}^{N-1} \frac{d^2\Phi}{dx^2}(\xi_k^N)(K_{t_{k+1}^N} - K_{t_k^N})^2 \le \frac{1}{2} M \sum_{k=0}^{N-1} (K_{t_{k+1}^N} - K_{t_k^N})^2 \to 0,\ q.s., \text{ as } N \to +\infty.$$

The proof is complete. □

Sketch of the proof of Lemma 1.29: To combine the result above with the ones in Peng [75], we decompose X into $M^X + K$, where M^X denotes the G-Itô part of X. Given a sequence of refining partitions $\{\pi_{[s,t]}^{2^N}\}_{N\in\mathbb{N}}$: for each $N \in \mathbb{N}$,

$$\pi_{[s,t]}^{2^N} = \{t_0^{2^N}, t_1^{2^N} \dots, t_{2^N}^{2^N}\} = \{s, s+\delta, \dots, s + 2^N\delta = t\},$$

we have from the second order Taylor expansion

$$
\begin{aligned}
\Phi(X_t)-\Phi(X_s) &= \sum_{k=0}^{2^N-1}(\Phi(X_{t_{k+1}^{2^N}})-\Phi(X_{t_{k+1}^{2^N}}))\\
&= \sum_{k=0}^{2^N-1}\frac{d\Phi}{dx}(X_{t_k^{2^N}})(M^X_{t_{k+1}^{2^N}}-M^X_{t_k^{2^N}})+\frac{1}{2}\sum_{k=0}^{2^N-1}\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})(M^X_{t_{k+1}^{2^N}}-M^X_{t_k^{2^N}})^2\\
&+\sum_{k=0}^{2^N-1}\frac{d^2\Phi}{dx^2}(\xi_k^{2^N})(M^X_{t_{k+1}^{2^N}}-M^X_{t_k^{2^N}})(K_{t_{k+1}^{2^N}}-K_{t_k^{2^N}})\\
&+\frac{1}{2}\sum_{k=0}^{2^N-1}\frac{d^2\Phi}{dx^2}(\xi_k^{2^N})(K_{t_{k+1}^{2^N}}-K_{t_k^{2^N}})^2\\
&+\frac{1}{2}\sum_{k=0}^{2^N-1}\left(\frac{d^2\Phi}{dx^2}(\xi_k^{2^N})-\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\right)(M^X_{t_{k+1}^{2^N}}-M^X_{t_k^{2^N}})^2\\
&+\sum_{k=0}^{2^N-1}\frac{d\Phi}{dx}(X_{t_k^{2^N}})(K_{t_{k+1}^{2^N}}-K_{t_k^{2^N}})\\
&= I_1^N+I_2^N+I_3^N+I_4^N+I_5^N+I_6^N,
\end{aligned}
$$

where $\xi_k^{2^N}$ satisfies $X_{t_k^{2^N}}\wedge X_{t_{k+1}^{2^N}}\le \xi_k^{2^N}\le X_{t_k^{2^N}}\vee X_{t_{k+1}^{2^N}}$, q.s..

A key point in the proof is to verify the following convergence results in $\bar{M}_G^2([0,T])$:

$$
\sum_{k=0}^{2^N-1}\frac{d\Phi}{dx}(X_{t_k^{2^N}})\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(\cdot)\to\frac{d\Phi}{dx}(X_\cdot),\ \text{as } N\to+\infty; \tag{1.9}
$$

and

$$
\sum_{k=0}^{2^N-1}\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(\cdot)\to\frac{d^2\Phi}{dx^2}(X_\cdot),\ \text{as } N\to+\infty. \tag{1.10}
$$

For the G-Itô part M^X, we deduce by the BDG type inequalities

$$
\int_s^t\bar{\mathbb{E}}\Big[\Big|\sum_{k=0}^{2^N-1}M^X_{t_k^{2^N}}\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(u)-M^X\Big|^2\Big]du\le M|t-s|(\delta+\delta^2)\to 0,\ \text{as } N\to+\infty. \tag{1.11}
$$

For the increasing process K, thanks to assumption (1.7), for each $u\in[s,t]$,

$$
\lim_{N\to+\infty}\bar{\mathbb{E}}\Big[\Big|\sum_{k=0}^{2^N-1}K_{t_k^{2^N}}\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(u)-K_u\Big|^2\Big]=0. \tag{1.12}
$$

Moreover,

$$
\int_s^t\bar{\mathbb{E}}\Big[\Big|\sum_{k=0}^{2^N-1}K_{t_k^{2^N}}\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(u)\Big|^2\Big]du\le\int_s^t\bar{\mathbb{E}}[K_u^2]du<+\infty.
$$

By Lebesgue's dominated convergence theorem to the integral on $[s,t]$, we deduce

$$
\lim_{N\to+\infty}\int_s^t\bar{\mathbb{E}}\Big[\Big|\sum_{k=0}^{2^N-1}K_{t_k^{2^N}}\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(u)-K_u\Big|^2\Big]du=0. \tag{1.13}
$$

Combining (1.11) and (1.13), (1.9) and (1.10) are readily obtained by the Lipschitz continuity of $\frac{d\Phi}{dx}$ and $\frac{d^2\Phi}{dx^2}$. Then, we can proceed similarly to Peng [75] to treat with I_1^N and I_2^N.

On the other hand, due to the boundedness of $\frac{d^2\Phi}{dx^2}$ and the boundedness and uniform continuity of paths $M_\cdot^X(\omega)$ and $K_\cdot(\omega)$ on $[0,T]$, for each $\omega \in A^c$, we can easily obtain that I_3^N and I_4^N q.s. vanish.

For I_5^N, we calculate

$$|I_5^N| \leq \frac{C}{2}\sum_{k=0}^{2^N-1}|\xi_k^{2^N} - X_{t_k^{2^N}}||M_{t_{k+1}^{2^N}}^X - M_{t_k^{2^N}}^X|^2$$
$$\leq \frac{C}{2}\Bigg(\sum_{k=0}^{2^N-1}|(\xi^1)_k^{2^N} - M_{t_k^{2^N}}^X||M_{t_{k+1}^{2^N}}^X - M_{t_k^{2^N}}^X|^2 + \sum_{k=0}^{2^N-1}|(\xi^2)_k^{2^N} - K_{t_k^{2^N}}||M_{t_{k+1}^{2^N}}^X - M_{t_k^{2^N}}^X|^2\Bigg),$$

where $(\xi^1)_k^{2^N}$ satisfies $M_{t_k^{2^N}}^X \wedge M_{t_{k+1}^{2^N}}^X \leq (\xi^1)_k^{2^N} \leq M_{t_k^{2^N}}^X \vee M_{t_{k+1}^{2^N}}^X$ and $(\xi^2)_k^{2^N}$ satisfies $K_{t_k^{2^N}} \leq (\xi^2)_k^{2^N} \leq K_{t_{k+1}^{2^N}}$, q.s.. The result in Peng [75] shows that the first part converges to 0 in $\bar{M}_G^2([0,T])$, whereas the second part vanishes as a result of the uniform continuity of paths $K_\cdot(\omega)$ on $[0,T]$, for all $\omega \in A^c$ and the q.s. boundedness of the quadratic variation of the G-Itô part M^X.

For I_6^N, it converges to $\int_s^t \frac{d\Phi}{dx}(X_u)dK_u$, q.s. by Definition 1.20. □

Remark 1.31 *In the proof of the classical Itô's formula, (1.9) and (1.10) can be verified directly by the pathwise continuity of X and Lebesgue's dominated convergence theorem on the product space $[s,t]\times\Omega$. But in the G-framework, we lack such a theorem. In general, given an $X \in \bar{M}_G^2([0,T])$, the sequence of step processes*

$$\left\{\sum_{k=0}^{2^N-1} X_{t_k^{2^N}}\mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(\cdot)\right\}_{N\in\mathbb{N}}$$

could not converge to X in the sense of $\bar{M}_G^2([0,T])$. Thus, (1.7) is needed to ensure that (1.12) holds true.

In fact, the left-hand side of (1.8), particularly the term $\int_s^t \frac{d\Phi}{dx}(X_u)dK_u$, still belongs to $L_G^2(\Omega_t)$. A sufficient condition of this result is that $K_t \in L_G^2(\Omega_t)$, which can be verified by choosing a sequence such that $t_n \to t$ and for all $n \in \mathbb{N}$, $X_{t_n} \in L_G^2(\Omega_{t_n})$ (Remark 1.11 ensures the existence of this sequence) and by deduction from assumption (1.7).

Similar to Theorem 6.5 in Chapter III of Peng [75], we can extend G-Itô's formula in Lemma 1.29 to those Φ whose second derivatives $\frac{d^2\Phi}{dx^2}$ have polynomial growth. Unfortunately, this extension is at the cost of more restrictions on the increasing process K.

Theorem 1.32 *Let $\Phi \in \mathcal{C}^2(\mathbb{R})$ be a real function such that $\frac{d^2\Phi}{dx^2}$ satisfies a polynomial growth condition. Let f, h and g be bounded processes in $\bar{M}_G^2([0,T])$ and $K \in M_I([0,T]) \cap \bar{M}_G^2([0,T])$ satisfies that for each $t \in [0,T]$,*

$$\lim_{s\to t}\bar{\mathbb{E}}[|K_t - K_s|^2] = 0;$$

and for any $p > 2$, $\bar{\mathbb{E}}[K_T^p] < +\infty$. Then,

$$\begin{aligned}\Phi(X_t) - \Phi(X_s) = & \int_s^t \frac{d\Phi}{dx}(X_u)f_u du + \int_s^t \frac{d\Phi}{dx}(X_u)h_u d\langle B\rangle_u \\ & + \int_s^t \frac{d\Phi}{dx}(X_u)g_u dB_u + \int_s^t \frac{d\Phi}{dx}(X_u)dK_u \\ & + \frac{1}{2}\int_s^t \frac{d^2\Phi}{dx^2}(X_u)g_u^2 d\langle B\rangle_u,\ q.s..\end{aligned} \tag{1.14}$$

Proof: By the same argument in the proof of Theorem 6.5 of Peng [75], we can choose a sequence of functions $\Phi^N \in \mathcal{C}_0^2(\mathbb{R})$, such that for each $x \in \mathbb{R}$,

$$|\Phi^N(x) - \Phi(x)| + \left|\frac{d\Phi^N}{dx}(x) - \frac{d\Phi}{dx}(x)\right| + \left|\frac{d^2\Phi^N}{dx^2}(x) - \frac{d^2\Phi}{dx^2}(x)\right| \leq \frac{C}{N}(1+|x|^k), \tag{1.15}$$

where C and k are positive constants independent of N. Obviously, Φ^N satisfies the conditions in Lemma 1.29. Therefore,

$$\begin{aligned}\Phi^N(X_t)-\Phi^N(X_s)=&\int_s^t\frac{d\Phi^N}{dx}(X_u)f_udu+\int_s^t\frac{d\Phi^N}{dx}(X_u)h_ud\langle B\rangle_u\\&+\int_s^t\frac{d\Phi^N}{dx}(X_u)g_udB_u+\int_s^t\frac{d\Phi^N}{dx}(X_u)dK_u\\&+\frac{1}{2}\int_s^t\frac{d^2\Phi^N}{dx^2}(X_u)g_u^2d\langle B\rangle_u.\end{aligned}\tag{1.16}$$

Borrowing the notation in the proof of Lemma 1.29 and using the BDG type inequalities, we have

$$\bar{\mathbb{E}}[\sup_{0\le t\le T}|X_t|^{2k}]\le C(\bar{\mathbb{E}}[\sup_{0\le t\le T}|M_t^X|^{2k}]+\bar{\mathbb{E}}[|K_T|^{2k}])<+\infty.\tag{1.17}$$

Then, from (1.15) and (1.17), we deduce that as $N\to+\infty$,

$$\begin{aligned}&\Phi^N(X_t)\to\Phi(X_t),\ \text{in } L_G^2(\Omega_t);\\&\frac{d\Phi^N}{dx}(X_\cdot)\to\frac{d\Phi}{dx}(X_\cdot),\ \text{in } \bar{M}_G^2([0,T]);\\&\frac{d^2\Phi^N}{dx^2}(X_\cdot)\to\frac{d^2\Phi}{dx^2}(X_\cdot),\ \text{in } \bar{M}_G^2([0,T]).\end{aligned}\tag{1.18}$$

We can proceed as in Peng [75] to show that the terms on the right-hand side of (1.16) converge to their corresponding terms in (1.14), except $\int_s^t\frac{d\Phi^N}{dx}(X_u)dK_u$. To complete the proof, it suffices to show that for each $\omega\in A^c$,

$$\begin{aligned}&\left|\int_s^t\frac{d\Phi^N}{dx}(X_u(\omega))dK_u(\omega)-\int_s^t\frac{d\Phi}{dx}(X_u(\omega))dK_u(\omega)\right|\\\le&\frac{C}{N}\int_s^t(1+|X_u(\omega)|^k)dK_u(\omega)\le\frac{C}{N}(1+M_\omega^k)K_T(\omega)\to0,\ \text{as } N\to+\infty,\end{aligned}$$

by the continuity and boundedness of paths $X_\cdot(\omega)$ and $K_\cdot(\omega)$ on $[0,T]$. □

Remark 1.33 *If $|\frac{d^2\Phi}{dx^2}(x)|\le C(1+|x|^k)$, for some $k\ge1$, then the condition on K could be weakened to $\bar{\mathbb{E}}[|K_T|^{2(k+3)}]<+\infty$.*

1.4 Reflected G-Brownian motion

Before moving to the main result of this chapter, we first consider a reduced RGSDE, that is, taking $f=h\equiv0$ and $g\equiv1$, only a G-Brownian motion and an increasing process drive the dynamic on the right-hand side of (1.1). In what follows, we establish the solvability to the RGSDE of this type, i.e., the existence and uniqueness of reflected G-Brownian Motion.

Let y be a real valued continuous function on $[0,T]$ with $y_0\ge0$. It is well-known that there exists a unique pair (x,k) of functions on $[0,T]$ such that $x=y+k$, where x is positive, k is an increasing and continuous function that starts from 0. Moreover, the Riemann-Stieltjes integral $\int_0^T x_tdk_t=0$. The solution to this Skorokhod problem on $[0,T]$ is given by

$$\begin{cases}x_t=y_t+k_t;\\k_t=\sup_{s\le t}x_s^-,\end{cases}\tag{1.19}$$

which is explicit and unique.

Theorem 1.34 *For any $p\ge1$, there exists a unique pair of processes (X,K) in $\bar{M}_G^p([0,T])\times(M_I([0,T])\cap\bar{M}_G^p([0,T]))$, such that*

$$X_t=B_t+K_t,\ 0\le t\le T,\ q.s.,\tag{1.20}$$

where (a) X is positive; (b) $K_0=0$; and (c) $\int_0^T X_tdK_t=0$, q.s..

Proof: With the help of (1.19), we define a pair of processes (X, K) pathwisely on $[0,T]$:

$$\begin{cases} X_t(\omega) = B_t(\omega) + K_t(\omega); \\ K_t(\omega) = \sup_{s\le t} B_s^-(\omega). \end{cases} \tag{1.21}$$

Obviously, $K \in M_I([0,T])$ and (a), (b) and (c) are satisfied. Therefore, to complete the proof, we need only verify that $K \in \bar{M}_G^p([0,T])$.

Because for all $1 \le p' < p$, $\bar{M}_G^{p'}([0,T]) \subset \bar{M}_G^p([0,T])$, we can assume that $p > 2$ without loss of generality. Given a sequence of partitions $\{\pi_{[0,T]}^N\}_{N\in\mathbb{N}}$, we set

$$(B_t^-)^N(\omega) := \sum_{k=0}^{N-1} B_{t_k^N}^-(\omega)\mathbf{1}_{[t_k^N, t_{k+1}^N)}(t),\ 0 \le t \le T;$$

and

$$\sup_{0\le s\le t}(B_s^-)^N := \sum_{k=0}^{N-1} \max_{l\in\{0,1,\dots,k\}} B_{t_l^N}^- \mathbf{1}_{[t_k^N, t_{k+1}^N)}(t),\ 0 \le t \le T.$$

We observe that both $((B_t^-)^N)_{0\le t\le T}$ and $(\sup_{0\le s\le t}(B_s^-)^N)_{0\le t\le T}$ are step processes in $\bar{M}_G^p([0,T])$. Because

$$\begin{aligned} &\bar{\mathbb{E}}[|\sup_{0\le s\le t}(B_s^-)^N - \sup_{0\le s\le t} B_s^-|^p] \le \bar{\mathbb{E}}[\sup_{0\le s\le t}|(B_s^-)^N - B_s^-|^p] \\ \le\ &\bar{\mathbb{E}}[\sup_{0\le t\le T}|B_t^N - B_t|^p] \le \bar{\mathbb{E}}[\sup_{k\in\mathbb{N}\cap[0,N)}\ \sup_{t_k\le t< t_{k+1}} |B_t - B_{t_k^N}|^p], \end{aligned}$$

letting $f = h \equiv 0$ and $g \equiv 1$ in (1.6), we obtain

$$\bar{\mathbb{E}}[|\sup_{0\le s\le t}(B_s^-)^N - \sup_{0\le s\le t} B_s^-|^p] \le C\mu(\pi_{[0,T]}^N)^{\frac{p}{2}-1} \to 0, \text{ as } N \to +\infty,$$

which shows that $(\sup_{0\le s\le t}(B_s^-)^N)_{0\le t\le T}$ converges to K in $\bar{M}_G^p([0,T])$.

On the other hand, the uniqueness of such a pair (X, K) is inherited from the solution to the Skorokhod problem pathwisely. The proof is complete. □

Remark 1.35 *We call the process X in Theorem 1.34 a G-reflected Brownian motion on the half-line $[0,+\infty)$.*

Furthermore, if the G-Brownian motion B is replaced by some G-Itô process, we have the following statement similar to Theorem 1.34:

Theorem 1.36 *For some $p > 2$, consider a q.s. continuous G-Itô process Y defined in the form of (1.5) whose coefficients are all elements in $\bar{M}_G^p([0,T])$. Then, there exists a unique pair of processes (X, K) in $\bar{M}_G^p([0,T]) \times (M_I([0,T]) \cap \bar{M}_G^p([0,T]))$ such that*

$$X_t = Y_t + K_t,\ 0 \le t \le T,\ q.s., \tag{1.22}$$

where (a) X is positive; (b) $K_0 = 0$; and (c) $\int_0^T X_t dK_t = 0$, q.s..

We omit the proof, because it is an analogue to the proof above and deduced mainly by the integrability of the coefficients of Y and (1.6).

1.5 Scalar valued reflected GSDEs

We state our main result in this section by giving the existence and uniqueness of the solutions to the scalar valued RGSDEs with Lipschitz coefficients. Additionally, a comparison theorem is given at the end of this chapter.

1.5.1 Formulation to reflected GSDEs

We consider the following scalar valued RGSDE:

$$X_t = x + \int_0^t f_s(X_s)ds + \int_0^t h_s(X_s)d\langle B\rangle_s + \int_0^t g_s(X_s)dB_s + K_t,\ 0 \le t \le T,\ q.s., \tag{1.23}$$

where

(A1) The initial condition $x \in \mathbb{R}$;

(A2) For some $p > 2$, the coefficients $f, h, g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are given functions that satisfy for each $x \in \mathbb{R}$, $f_.(x)$, $h_.(x)$, $g_.(x) \in \bar{M}_G^p([0,T])$;

(A3) The coefficients f, h and g satisfy a Lipschitz condition, i.e., for each $t \in [0,T]$ and $x, x' \in \mathbb{R}$, $|f_t(x) - f_t(x')| + |h_t(x) - h_t(x')| + |g_t(x) - g_t(x')| \le C_L|x - x'|$, q.s.;

(A4) The obstacle S is a G-Itô process whose coefficients are all elements in $\bar{M}_G^p([0,T])$, and we shall always assume that $S_0 \le x$, q.s..

The solution to the RGSDE (1.23) is a pair of processes (X,K) that take values both in $\mathbb{R}$ and satisfy:

(i) $X \in \bar{M}_G^p([0,T])$ and $X_t \ge S_t$, $0 \le t \le T$, q.s.;

(ii) $K \in M_I([0,T]) \cap \bar{M}_G^p([0,T])$ and $K_0 = 0$, q.s.;

(iii) $\int_0^T (X_t - S_t)dK_t = 0$, q.s..

1.5.2 Some a priori estimates and uniqueness result

Let (X,K) be a solution to (1.23). Replacing Y_t by $x+\int_0^t f_s(X_s)ds+\int_0^t h_s(X_s)d\langle B\rangle_s+\int_0^t g_s(X_s)dB_s-S_t$ and X_t by $X_t - S_t$ in (1.22), we have the following representation of K:

$$\begin{aligned} K_t = \sup_{0\le s\le t}\Bigg(x + \int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d\langle B\rangle_u \\ + \int_0^s g_u(X_u)dB_u - S_s\Bigg)^-,\ 0 \le t \le T,\ q.s.. \end{aligned} \tag{1.24}$$

We now give an a priori estimate on the uniform norm of the solution.

Proposition 1.37 *Let (X,K) be a solution to (1.23). Then, there exists a constant $C > 0$ such that*

$$\begin{aligned} \bar{\mathbb{E}}[\sup_{0\le t\le T}|X_t|^p] + \bar{\mathbb{E}}[K_T^p] \le C\Bigg(|x|^p + \int_0^T (\bar{\mathbb{E}}[|f_t(0)|^p] \\ + \bar{\mathbb{E}}[|h_t(0)|^p] + \bar{\mathbb{E}}[|g_t(0)|^p])dt + \bar{\mathbb{E}}[\sup_{0\le t\le T}|S_t^+|^p]\Bigg). \end{aligned}$$

Proof: As X is the solution to (1.23), we obtain

$$\begin{aligned} \bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s|^p] &\le \bar{\mathbb{E}}[\sup_{0\le s\le t}|x + \textstyle\int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d\langle B\rangle_u + \int_0^s g_u(X_u)dB_u + K_s|^p] \\ &\le C(|x|^p + \bar{\mathbb{E}}[\sup_{0\le s\le t}|\textstyle\int_0^s f_u(X_u)du|^p] + \bar{\mathbb{E}}[\sup_{0\le s\le t}|\int_0^s h_u(X_u)d\langle B\rangle_u|^p] \\ &+ \bar{\mathbb{E}}[\sup_{0\le s\le t}|\textstyle\int_0^s g_u(X_u)dB_u|^p] + \bar{\mathbb{E}}[|K_t|^p]). \end{aligned} \tag{1.25}$$

In a similar way to (1.25), from the representation of K (1.24), we have

$$\begin{aligned} \bar{\mathbb{E}}[K_t^p] &\le \bar{\mathbb{E}}[\sup_{0\le s\le t}((x + \textstyle\int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d\langle B\rangle_u + \int_0^s g_u(X_u)dB_u - S_s)^-)^p] \\ &\le \bar{\mathbb{E}}[\sup_{0\le s\le t}((x + \textstyle\int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d\langle B\rangle_u + \int_0^s g_u(X_u)dB_u - S_s^+)^-)^p] \\ &\le \bar{\mathbb{E}}[\sup_{0\le s\le t}|x + \textstyle\int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d\langle B\rangle_u + \int_0^s g_u(X_u)dB_u - S_s^+|^p] \\ &\le C(|x|^p + \bar{\mathbb{E}}[\sup_{0\le s\le t}|\textstyle\int_0^s f_u(X_u)du|^p] + \bar{\mathbb{E}}[\sup_{0\le s\le t}|\int_0^s h_u(X_u)d\langle B\rangle_u|^p] \\ &+ \bar{\mathbb{E}}[\sup_{0\le s\le t}|\textstyle\int_0^s g_u(X_u)dB_u|^p] + \bar{\mathbb{E}}[\sup_{0\le s\le t}|S_s^+|^p]). \end{aligned} \tag{1.26}$$

Combining (1.25) and (1.26) and applying BDG type inequalities, we get

$$\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s|^p]+\bar{\mathbb{E}}[K_t^p]\le C(|x|^p+\int_0^t(\bar{\mathbb{E}}[|f_s(X_s)|^p] \\ +\bar{\mathbb{E}}[|h_s(X_s)|^p]+\bar{\mathbb{E}}[|g_s(X_s)|^p])ds+\bar{\mathbb{E}}[\sup_{0\le s\le t}|S_s^+|^p].$$

By assumption (A3), we calculate

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s|^p]+\bar{\mathbb{E}}[K_t^p] &\le C(|x|^p+\int_0^t(\bar{\mathbb{E}}[(|f_s(0)|+C_L|X_s|)^p]+\bar{\mathbb{E}}[(|h_s(0)|+C_L|X_s|)^p]\\ &\quad+\bar{\mathbb{E}}[(|g_s(0)+C_L|X_s|)^p])ds+\bar{\mathbb{E}}[\sup_{0\le s\le t}|S_s^+|^p]\\ &\le C(|x|^p+\int_0^t(\bar{\mathbb{E}}[|f_s(0)|^p]+\bar{\mathbb{E}}[|h_s(0)|^p]+\bar{\mathbb{E}}[|g_s(0)|^p])ds\\ &\quad+\bar{\mathbb{E}}[\sup_{0\le s\le t}|S_s^+|^p]+\int_0^t\bar{\mathbb{E}}[|X_s|^p]ds)\\ &\le C(|x|^p+\int_0^T(\bar{\mathbb{E}}[|f_t(0)|^p]+\bar{\mathbb{E}}[|h_t(0)|^p]+\bar{\mathbb{E}}[|g_t(0)|^p])dt\\ &\quad+\bar{\mathbb{E}}[\sup_{0\le t\le T}|S_t^+|^p]+\int_0^t\bar{\mathbb{E}}[\sup_{0\le u\le s}|X_u|^p]ds).\end{aligned} \tag{1.27}$$

Applying Gronwall's lemma to $\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s|^p]$, we deduce

$$\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_t|^p]\le C\Bigg(|x|^p+\int_0^T(\bar{\mathbb{E}}[|f_t(0)|^p]+\bar{\mathbb{E}}[|h_t(0)|^p] \\ +\bar{\mathbb{E}}[|g_t(0)|^p])dt+\bar{\mathbb{E}}[\sup_{0\le t\le T}|S_t^+|^p]\Bigg),\ 0\le t\le T. \tag{1.28}$$

Putting (1.28) into (1.27), the result follows. □

In the following theorem, we estimate the variation in the solutions induced by a variation in the coefficients and the obstacle processes.

Theorem 1.38 *Let (x^1,f^1,h^1,g^1,S^1) and (x^2,f^2,h^2,g^2,S^2) be two sets of coefficients that satisfy the assumptions (A1)-(A4) and (X^i,K^i) the solution to the RGSDE corresponding to (x^i,f^i,h^i,g^i,S^i), $i=1,2$. Define*

$$\Delta x:=x^1-x^2,\ \Delta f:=f^1-f^2,\Delta h:=h^1-h^2,\ \Delta g:=g^1-g^2;\\ \Delta S:=S^1-S^2,\ \Delta X:=X^1-X^2,\ \Delta K:=K^1-K^2.$$

Then, there exists a constant $C>0$ such that

$$\bar{\mathbb{E}}[\sup_{0\le t\le T}|\Delta X_t|^p]\le C\Bigg(|\Delta x|^p+\int_0^T(\bar{\mathbb{E}}[|\Delta f_t(X_t^1)|^p]+\bar{\mathbb{E}}[|\Delta h_t(X_t^1)|^p] \\ +\bar{\mathbb{E}}[|\Delta g_t(X_t^1)|^p])dt+\bar{\mathbb{E}}[\sup_{0\le t\le T}|\Delta S_t|^p]\Bigg).$$

Proof: Defining

$$(M^X)_t^i:=x^i+\int_0^t f_s^i(X_s^i)ds+\int_0^t h_s^i(X_s^i)d\langle B\rangle_s+\int_0^t g_s^i(X_s^i)dB_s,\ 0\le t\le T,\ i=1,2;$$

and

$$\Delta M^X:=(M^X)^1-(M^X)^2,$$

we calculate in a similar way to the proof of Proposition 1.37

$$\begin{aligned}
\bar{\mathbb{E}}[\sup_{0\le s\le t}|(\Delta M^X)_s|^p] &\le \bar{\mathbb{E}}[\sup_{0\le s\le t}|\Delta x + \textstyle\int_0^s (f_u^1(X_u^1) - f_u^2(X_u^2))du \\
&\quad + \textstyle\int_0^s (h_u^1(X_u^1) - h_u^2(X_u^2))d\langle B\rangle_u + \int_0^s (g_u^1(X_u^1) - g_u^2(X_u^2))dB_u|^p] \\
&\le \bar{\mathbb{E}}[\sup_{0\le s\le t}|\Delta x + \textstyle\int_0^s \Delta f_u(X_u^1)du + \int_0^s (f_u^2(X_u^1) - f_u^2(X_u^2))du \\
&\quad + \textstyle\int_0^s \Delta h_u(X_u^1)d\langle B\rangle_u + \int_0^s (h_u^2(X_u^1) - h_u^2(X_u^2))d\langle B\rangle_u \\
&\quad + \textstyle\int_0^s \Delta g_u(X_u^1)dB_u + \int_0^s (g_u^2(X_u^1) - g_u^2(X_u^2))dB_u|^p] \\
&\le C(|\Delta x|^p + \textstyle\int_0^t (\bar{\mathbb{E}}[|\Delta f_s(X_s^1)|^p] + \bar{\mathbb{E}}[|\Delta h_s(X_s^1)|^p] \\
&\quad + \bar{\mathbb{E}}[|\Delta g_s(X_s^1)|^p])ds + \textstyle\int_0^t \bar{\mathbb{E}}[|\Delta X_s|^p]ds)
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathbb{E}}[\sup_{0\le s\le t}|\Delta K_s|^p] &= \bar{\mathbb{E}}[\sup_{0\le s\le t}|\sup_{0\le u\le s}((M^X)_u^1 - S_u^1)^- - \sup_{0\le u\le s}((M^X)_u^2 - S_u^2)^-|^p] \\
&\le \bar{\mathbb{E}}[\sup_{0\le s\le t}|\sup_{0\le u\le s}|((M^X)_u^1 - S_u^1)^- - ((M^X)_u^2 - S_u^2)^-||^p] \\
&= \bar{\mathbb{E}}[\sup_{0\le s\le t}|((M^X)_s^1 - S_s^1)^- - ((M^X)_s^2 - S_s^2)^-|^p] \\
&\le \bar{\mathbb{E}}[\sup_{0\le s\le t}|((M^X)_s^1 - S_s^1) - ((M^X)_s^2 - S_s^2)|^p] \\
&\le C(\bar{\mathbb{E}}[\sup_{0\le s\le t}|\Delta(M^X)_s|^p] + \bar{\mathbb{E}}[\sup_{0\le s\le t}|\Delta S_s|^p]).
\end{aligned} \tag{1.29}$$

Then, we have

$$\begin{aligned}
\bar{\mathbb{E}}[\sup_{0\le s\le t}|\Delta X_s|^p] &\le \bar{\mathbb{E}}[\sup_{0\le s\le t}|(\Delta M^X)_s + \Delta K_s|^p|] \\
&\le C(\bar{\mathbb{E}}[\sup_{0\le s\le t}|(\Delta M^X)_s|^p] + \bar{\mathbb{E}}[\sup_{0\le s\le t}|\Delta K_s|^p]) \\
&\le C(|\Delta x|^p + \textstyle\int_0^t (\bar{\mathbb{E}}[|\Delta f_s(X_s^1)|^p] + \bar{\mathbb{E}}[|\Delta h_s(X_s^1)|^p] \\
&\quad + \bar{\mathbb{E}}[|\Delta g_s(X_s^1)|^p])ds + \bar{\mathbb{E}}[\sup_{0\le s\le t}|\Delta S_s|^p] + \textstyle\int_0^t \bar{\mathbb{E}}[|\Delta X_s|^p]ds). \\
&\le C(|\Delta x|^p + \textstyle\int_0^T (\bar{\mathbb{E}}[|\Delta f_t(X_t^1)|^p] + \bar{\mathbb{E}}[|\Delta h_t(X_t^1)|^p] + \bar{\mathbb{E}}[|\Delta g_t(X_t^1)|^p])dt \\
&\quad + \bar{\mathbb{E}}[\sup_{0\le t\le T}|\Delta S_t|^p] + \textstyle\int_0^t \bar{\mathbb{E}}[\sup_{0\le u\le s}|\Delta X_u|^p]ds).
\end{aligned}$$

Gronwall's lemma gives the desired result. □

We deduce immediately the following uniqueness result by taking $x^1 = x^2$, $f^1 = f^2$, $h^1 = h^2$, $g^1 = g^2$ and $S^1 = S^2$ in Theorem 1.38.

Theorem 1.39 *Under assumptions (A1)-(A4), there exists at most one solution in $\bar{M}_G^p([0,T]) \times (M_I([0,T]) \cap \bar{M}_G^p([0,T]))$ to the RGSDE (1.23).*

1.5.3 Existence result

We now turn to the following existence result for the RGSDE (1.23). The proof will be based on a Picard iteration.

Theorem 1.40 *Under assumptions (A1)-(A4), there exists a unique solution in $\bar{M}_G^p([0,T]) \times (M_I([0,T]) \cap \bar{M}_G^p([0,T]))$ to the RGSDE (1.23).*

Proof: We set $X^0 = x$ and $K^0 = 0$. For each $n \in \mathbb{N}^+$, X^{n+1} is given by recurrence:

$$X_t^{n+1} = x + \int_0^t f_s(X_s^n)ds + \int_0^t h_s(X_s^n)d\langle B\rangle_s + \int_0^t g_s(X_s^n)dB_s + K_t^{n+1},\ 0 \le t \le T, \tag{1.30}$$

where

(a) $X^{n+1} \in \bar{M}_G^p([0,T]),\ X_t^{n+1} \geq S_t,\ 0 \leq t \leq T,\ q.s.$;

(b) $K^{n+1} \in M_I([0,T]) \cap \bar{M}_G^p([0,T]),\ K_0^{n+1} = 0,\ q.s.$;

(c) $\int_0^T (X_t^{n+1} - S_t) dK_t^{n+1} = 0,\ q.s.$.

Substituting X^{n+1} by $\tilde{X}^{n+1} + S_t$ on the left-hand side of (1.30), we know that (X^{n+1}, K^{n+1}) is well defined in $\bar{M}_G^p([0,T]) \times (M_I([0,T]) \cap \bar{M}_G^p([0,T]))$ by Theorem 1.36.

First, we establish an a priori estimate uniform in n for $\{\bar{\mathbb{E}}[\sup_{0\leq t\leq T} |X_t^n|^p]\}_{n\in\mathbb{N}}$. In a similar way to (1.27), we have

$$\bar{\mathbb{E}}[\sup_{0\leq s\leq t} |X_t^{n+1}|^p] \leq C\bigg(|x|^p + \int_0^T (\bar{\mathbb{E}}[|f_t(0)|^p] + \bar{\mathbb{E}}[|h_t(0)|^p] + \bar{\mathbb{E}}[|g_t(0)|^p])dt + \bar{\mathbb{E}}[\sup_{0\leq t\leq T} |S_t^+|^p] + \int_0^t \bar{\mathbb{E}}[\sup_{0\leq u\leq s} |X_u^n|^p] ds\bigg).$$

By recurrence, it is easy to verify that for all $n \in \mathbb{N}$,

$$\bar{\mathbb{E}}[\sup_{0\leq s\leq t} |X_s^n|^p] \leq p(t),\ 0 \leq t \leq T,$$

where $p(\cdot)$ is the solution to the following ordinary differential equation:

$$p(t) = C\bigg(|x|^p + \int_0^T (\bar{\mathbb{E}}[|f_t(0)|^p] + \bar{\mathbb{E}}[|h_t(0)|^p] + \bar{\mathbb{E}}[|g_t(0)|^p])dt + \bar{\mathbb{E}}[\sup_{0\leq t\leq T} |S_t^+|^p] + \int_0^t p(s) ds\bigg),$$

and $p(\cdot)$ is continuous and thus, bounded on $[0,T]$.

Secondly, for each n and $m \in \mathbb{N}$, we define

$$u_t^{n+1,m} := \bar{\mathbb{E}}[\sup_{0\leq s\leq t} |X_s^{n+m+1} - X_s^{n+1}|^p],\ 0 \leq t \leq T.$$

Following the procedures in the proof of Theorem 1.38, we have

$$u_t^{n+1,m} \leq C \int_0^t u_s^{n,m} ds.$$

Set

$$v_t^n := \sup_{m\in\mathbb{N}} u_t^{n,m},\ 0 \leq t \leq T,$$

then

$$0 \leq u_t^{n+1,m} \leq C \sup_{m\in\mathbb{N}} \int_0^t u_s^{n,m} ds \leq C \int_0^t \sup_{m\in\mathbb{N}} u_s^{n,m} ds = C \int_0^t v_s^n ds.$$

Taking the supremum over all $m \in \mathbb{N}$ on the left-hand side, we obtain

$$0 \leq v_t^{n+1} = \sup_{m\in\mathbb{N}} u_t^{n+1,m} \leq C \int_0^t v_s^n ds.$$

Finally, we define

$$\alpha_t := \limsup_{n\to+\infty} v_t^n,\ 0 \leq t \leq T.$$

It is easy to find that $v_t^n \leq Cp(t)$, where C is independent of n. By the Fatou-Lebesgue theorem, we have

$$0 \leq \alpha_t \leq C \int_0^t \alpha_s ds.$$

Gronwall's lemma gives

$$\alpha_t = 0,\ 0 \leq t \leq T,$$

which implies that $\{X^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence under the norm $(\mathbb{E}[\sup_{0\leq t\leq T}|\cdot|^p])^{\frac{1}{p}}$, whose limit is certainly in $\bar{M}_G^p([0,T])$. We denote the limit by X and set

$$K_t := \sup_{0\leq s\leq t}\left(x+\int_0^s f_u(X_u)du+\int_0^s h_u(X_u)d\langle B\rangle_u+\int_0^s g_u(X_u)dB_u-S_s\right)^-,\ 0\leq t\leq T.$$

Obviously, the pair of processes (X,K) satisfies (i) - (iii). We notice that

$$\bar{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\int_0^t (f_s(X_s^n)-f_s(X_s))ds\right|^p\right]\leq C\int_0^T \bar{\mathbb{E}}[|X_t^n-X_t|^p]dt;$$
$$\bar{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\int_0^t (h_s(X_s^n)-h_s(X_s))d\langle B\rangle_s\right|^p\right]\leq C\int_0^T \bar{\mathbb{E}}[|X_t^n-X_t|^p]dt;$$
$$\bar{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\int_0^t (g_s(X_s^n)-g_s(X_s))dB_s\right|^p\right]\leq C\int_0^T \bar{\mathbb{E}}[|X_t^n-X_t|^p]dt.$$

Then, one can verify that K^n converges to K in $\bar{M}_G^p([0,T])$ following the steps of (1.29). We conclude that the pair of processes (X,K), well defined in $\bar{M}_G^p([0,T])\times(M_I([0,T])\cap\bar{M}_G^p([0,T]))$, is a solution to the RGSDE (1.23). □

Remark 1.41 *Unlike a classical RSDE or RBSDE, the constraint process S here is assumed to be a G-Itô process instead of a continuous process with $\bar{\mathbb{E}}[\sup_{0\leq t\leq T}(S_t^+)^2]\leq+\infty$ (cf. El Karoui et al. [20]). In fact, this is a sufficient condition to ensure that K^{n+1} is still a $\bar{M}_G^p([0,T])$ process in (1.30) by Theorem 1.36, which may be weakened to:*

$$\bar{\mathbb{E}}[\sup_{s\leq u\leq t}|S_u-S_s|^p]\leq C|t-s|^{\frac{p}{2}}.$$

Remark 1.42 *We could also consider (1.23) with less regularity assumptions on the coefficients f, h, g and the obstacle S under a family $\mathcal{P}_W$ of local martingale measures by using the approach introduced in Soner et al. [88, 86, 87]. The only problem in this case is the aggregation of the processes in the Picard iteration (1.30) for the proof of existence. Adapting to the assumptions of Theorem 2.2 in Nutz [67], we assume in addition that we work under Zermelo-Fraenkel set theory with the axiom of choice and the continuum hypothesis, then the stochastic integral of Itô's type $\int_0^t g_s(X_s^n)dB_s$ can be well aggregated at each step of the recurrence. Thus, we can define from (1.19) a universal pair (X^{n+1},K^{n+1}) to make (1.30) $\mathbb{P}$-a.s. hold for all $\mathbb{P}\in\mathcal{P}_W$. Following the argument in the proof of Theorem 1.40 under each $\mathbb{P}\in\mathcal{P}_W$, there exists a pair $(X^{\mathbb{P}},K^{\mathbb{P}})$, such that (1.23) holds true $\mathbb{P}$-a.s. and (X^n,K^n) converges to $(X^{\mathbb{P}},K^{\mathbb{P}})$ uniformly in probability measure $\mathbb{P}$. By Lemma 2.5 in Nutz [67], there exists a universal pair (X,K) such that $(X,K)=(X^{\mathbb{P}},K^{\mathbb{P}})$, $\mathbb{P}$-a.s., which solves (1.23) under each $\mathbb{P}\in\mathcal{P}_W$ and thus, in the (weaker) sense of $\mathcal{P}_W$-q.s..*

Remark 1.43 *In contrast with the fact mentioned in Remark 3.3 of Matoussi et al. [61], our results can be directly applied to the symmetrical problem, i.e., the RGSDE with an upper barrier. This conclusion is because the proof is only based on a pathwise construction and a fixed-point iteration.*

1.5.4 Comparison principle

In this subsection, we establish a comparison principle for RGSDEs. At first, we assume additionally a bounded condition on the coefficients f, h and g and the obstacle process S, and then we remove it in the second step.

Theorem 1.44 *Given two RGSDEs that satisfy the assumptions (A1)-(A4), we additionally suppose in the following:*

(1) *$x^1\leq x^2$;*

(2) *f^i, h^i and $g^1=g^2=g$ are bounded and S^i are uniformly bounded from above, $i=1,2$;*

(3) *for each $x\in\mathbb{R}$, $f_t^1(x)\leq f_t^2(x)$, $h_t^1(x)\leq h_t^2(x)$; and $S_t^1\leq S_t^2$, $0\leq t\leq T$, q.s..*

Let (X^i, K^i) be a solution to the RGSDE with data (x^i, f^i, h^i, g, S^i), $i = 1, 2$, then

$$X_t^1 \le X_t^2,\ 0 \le t \le T,\ q.s..$$

Proof: Since f^i, h^i and g are bounded and S^i are uniformly bounded from above, $i = 1, 2$, using the BDG type inequalities to (1.24), we deduce that K_T^i has the moment for any arbitrarily large order and for each $t \in [0, T]$, $\lim_{s\to t} \bar{\mathbb{E}}[|K_t^i - K_s^i|^2] = 0$, $i = 1, 2$.

Notice that $(x^+)^2$ is not a $\mathcal{C}^2(\mathbb{R})$ function. We have to consider $(x^+)^3$ and apply the extended G-Itô's formula to $((X_t^1 - X_t^2)^+)^3$, then

$$\begin{aligned}
((X_t^1 - X_t^2)^+)^3 = &\, 3\int_0^t |(X_s^1 - X_s^2)^+|^2 (f_s^1(X_s^1) - f_s^2(X_s^2))ds \\
&+ 3\int_0^t |(X_s^1 - X_s^2)^+|^2 (h_s^1(X_s^1) - h_s^2(X_s^2))d\langle B\rangle_s \\
&+ 3\int_0^t |(X_s^1 - X_s^2)^+|^2 (g_s(X_s^1) - g_s(X_s^2))dB_s \\
&+ 3\int_0^t |(X_s^1 - X_s^2)^+|^2 d(K_s^1 - K_s^2) \\
&+ 3\int_0^t (X_s^1 - X_s^2)^+ |g_s(X_s^1) - g_s(X_s^2)|^2 d\langle B\rangle_s.
\end{aligned} \tag{1.31}$$

As on $\{X_t^1 > X_t^2\}$, $X_t^1 > X_t^2 \ge S_t^2 \ge S_t^1$, we have

$$\begin{aligned}
\int_0^t |(X_s^1 - X_s^2)^+|^2 d(K_s^1 - K_s^2) &= \int_0^t |(X_s^1 - X_s^2)^+|^2 dK_s^1 - \int_0^t |(X_s^1 - X_s^2)^+|^2 dK_s^2 \\
&\le \int_0^t |(X_s^1 - S_s^1)^+|^2 dK_s^1 - \int_0^t |(X_s^1 - X_s^2)^+|^2 dK_s^2 \\
&\le -\int_0^t |(X_s^1 - X_s^2)^+|^2 dK_s^2 \le 0,\ q.s..
\end{aligned} \tag{1.32}$$

We put (1.32) into (1.31) and then, by Lipschitz condition (A3) and by taking G-expectation on both sides of (1.32), we conclude

$$\bar{\mathbb{E}}[((X_t^1 - X_t^2)^+)^3] \le C\bar{\mathbb{E}}\left[\int_0^t ((X_s^1 - X_s^2)^+)^3 ds\right] \le C\int_0^t \bar{\mathbb{E}}[((X_s^1 - X_s^2)^+)^3]ds.$$

Using Gronwall's lemma, it follows that $\bar{\mathbb{E}}[((X_t^1 - X_t^2)^+)^3] = 0$, which implies the result. □

Theorem 1.45 *Given two RGSDEs that satisfy the assumptions (A1)-(A4), we additionally suppose in the following:*

(1) $x^1 \le x^2$ *and* $g^1 = g^2 = g$;

(2) *for each* $x \in \mathbb{R}$, $f_t^1(x) \le f_t^2(x)$ *and* $h_t^1(x) \le h_t^2(x)$; *and* $S_t^1 \le S_t^2$, $0 \le t \le T$, *q.s..*

Let (X^i, K^i) be a solution to the RGSDE with data (x^i, f^i, h^i, g, S^i), $i = 1, 2$, then

$$X_t^1 \le X_t^2,\ 0 \le t \le T,\ q.s..$$

Proof: First, we define the truncated functions for the coefficients and the obstacle process: for each $N \in \mathbb{N}^+$, $\xi_t^N(x) := (-N \vee \xi_t(x)) \wedge N$, where ξ denote f^i, h^i, g and $x \in \mathbb{R}$; and $(S^i)_t^N = S_t^i \wedge N$, $0 \le t \le T$, $i = 1, 2$. It is easy to verify that the truncated coefficients and the obstacle processes satisfy (A2) and (A3). Moreover, the truncated functions keep the order of the coefficients and the obstacle processes, that is, for each $N \in \mathbb{N}^+$,

$$(f^1)_t^N(x) \le (f^2)_t^N(x) \text{ and } (h^1)_t^N(x) \le (h^2)_t^N(x), \text{ for all } x \in \mathbb{R},\ 0 \le t \le T,\ q.s.;$$

and

$$(S^1)_t^N \leq (S^2)_t^N,\ 0 \leq t \leq T,\ q.s..$$

Consider the following RGSDEs:

$$\begin{aligned}(X^i)_t^N = x + \int_0^t (f^i)_s^N((X^i)_s^N)ds + \int_0^t (h^i)_s^N((X^i)_s^N)d\langle B\rangle_s \\ + \int_0^t g_s^N((X^i)_s^N)dB_s + (K^i)_t^N,\ 0 \leq t \leq T,\ q.s.,\ i=1,2,\end{aligned}$$

under the following conditions:

(a) $(X^i)^N \in \bar{M}_G^p([0,T]),\ (X^i)_t^N \geq (S^i)_t^N,\ 0 \leq t \leq T,\ q.s.;$

(b) $(K^i)^N \in M_I([0,T]) \cap \bar{M}_G^p([0,T]),\ (K^i)_0^N = 0,\ q.s.;$

(c) $\int_0^T ((X^i)_t^N - (S^i)_t^N)d(K^i)_t^N = 0,\ q.s..$

By Theorem 1.44, it is readily observed that for each $N \in \mathbb{N}$,

$$(X^1)_t^N \leq (X^2)_t^N,\ 0 \leq t \leq T,\ q.s.. \tag{1.33}$$

Meanwhile, by Theorem 1.38, we have

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{0\leq s\leq t}|(X^i)_s^N - X_s^i|^p] \leq C(\textstyle\int_0^T(\bar{\mathbb{E}}[|(f^i)_t^N(X_t^i) - f_t^i(X_t^i)|^p] + \bar{\mathbb{E}}[|(h^i)_t^N(X_t^i) - h_t^i(X_t^i)|^p] \\ +\bar{\mathbb{E}}[|g_t^N(X_t^i) - g_t(X_t^i)|^p])dt + \bar{\mathbb{E}}[\sup_{0\leq t\leq T}|(S^i)_t^N - S_t^i|^p] \\ + \textstyle\int_0^t \bar{\mathbb{E}}[\sup_{0\leq u\leq s}|(X^i)_u^N - X_u^i|^p]ds).\end{aligned}$$

Applying again Gronwall's lemma, we obtain

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{0\leq t\leq T}|(X^i)_t^N - X_t^i|^p] \leq C\bigg(\int_0^T (\bar{\mathbb{E}}[|(f^i)^N(t,X_t^i) - f^i(t,X_t^i)|^p] + \bar{\mathbb{E}}[|(h^i)^N(t,X_t^i) - h^i(t,X_t^i)|^p] \\ + \bar{\mathbb{E}}[|g^N(t,X_t^i) - g(t,X_t^i)|^p])dt + \bar{\mathbb{E}}[\sup_{0\leq t\leq T}|(S^i)_t^N - S_t^i|^p]\bigg).\end{aligned}$$

For each $t \in [0,T]$, we calculate

$$\begin{aligned}\bar{\mathbb{E}}[|(f^i)_t^N(X_t^i) - f_t^i(X_t^i)|^p] &\leq \bar{\mathbb{E}}[|f_t^i(X_t^i)|^p \mathbf{1}_{|f_t^i(X_t^i)|>N}] \\ &\leq \bar{\mathbb{E}}[(|f_t^i(0)| + C_L|X_t^i|)^p \mathbf{1}_{(|f_t^i(0)|+C_L|X_t^i|)>N}] \\ &\leq C(\bar{\mathbb{E}}[|f_t^i(0)|^p \mathbf{1}_{|f_t^i(0)|>\frac{N}{2}}] + \bar{\mathbb{E}}[|X_t^i|^p \mathbf{1}_{|X_t^i|>\frac{N}{2}}]).\end{aligned}$$

Taking into consideration that $f_\cdot(0)$ and $X^i \in \bar{M}_G^p([0,T])$, from the argument in Remark 1.11, we know that $f_t(0)$ and $X_t^i \in L_G^p([0,T])$ for almost every $t \in [0,T]$. Therefore, letting $N \to +\infty$, we have

$$\bar{\mathbb{E}}[|(f^i)_t^N(X_t^i) - f_t^i(X_t^i)|^p] \to 0.$$

Similarly, we also obtain

$$\bar{\mathbb{E}}[|(h^i)_t^N(X_t^i) - h_t^i(X_t^i)|^p] \to 0;$$

and

$$\bar{\mathbb{E}}[|(g^i)_t^N(X_t^i) - g_t^i(X_t^i)|^p] \to 0.$$

Using Lebesgue's dominated convergence theorem to the integrals on $[0,T]$, it follows that

$$\begin{aligned}\lim_{N\to+\infty}\int_0^T (\bar{\mathbb{E}}[|(f^i)^N(t,X_t^i) - f^i(t,X_t^i)|^p] + \bar{\mathbb{E}}[|(h^i)^N(t,X_t^i) - h^i(t,X_t^i)|^p] \\ + \bar{\mathbb{E}}[|g^N(t,X_t^i) - g(t,X_t^i)|^p])dt = 0.\end{aligned} \tag{1.34}$$

On the other hand,

$$\bar{\mathbb{E}}[\sup_{0\le t\le T}|(S^i)_t^N - S_t^i|^p] \le \bar{\mathbb{E}}[\sup_{0\le t\le T}(|S_t^i|^p \mathbf{1}_{\{|S_t^i|>N\}})] \le \bar{\mathbb{E}}[\sup_{0\le t\le T}|S_t^i|^p \mathbf{1}_{\{\sup\limits_{0\le t\le T}|S_t^i|>N\}}].$$

By the proof of Theorem 1.36, we know that $\sup\limits_{0\le t\le T} S_t^i$ is an element in $L_G^p(\Omega_T)$. So we have

$$\bar{\mathbb{E}}[\sup_{0\le t\le T}|(S^i)_t^N - S_t^i|^p] \le \bar{\mathbb{E}}[\sup_{0\le t\le T}|S_t^i|^p \mathbf{1}_{\{\sup\limits_{0\le t\le T}|S_t^i|>N\}}] \to 0, \text{ as } N \to +\infty. \tag{1.35}$$

Combining (1.34) and (1.35), we obtain

$$\bar{\mathbb{E}}[\sup_{0\le t\le T}|(X^i)_t^N - X_t^i|^p] \to 0, \text{ as } N \to +\infty. \tag{1.36}$$

Then, (1.33) and (1.36) yield the desired result . □

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Chapter 2

Localization Methods for GSDEs

Abstract: In this chapter, stochastic integrals with respect to a bounded variation process are discussed, an extension of G-Itô's formula is proved and stochastic differential equations driven by G-Brownian motion (GSDEs) with local Lipschitz conditions are discussed on the space $M_w^p([0,T];\mathbb{R}^n)$ via a localization method.

Key words. G-Brownian motion; stopping times; bounded variation processes; G-Itô's formula; G-stochastic differential equations.

AMS subject classifications. 60H05; 60H10

2.1 Introduction

In the classical framework, the continuous-time stochastic calculus was first established on the Wiener space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, where Ω is a space that consists of all continuous paths starting from 0, $\mathcal{F}$ is the filtration generated by the canonical process B, and $\mathbb{P}_0$ is the Wiener measure, under which B is a Brownian motion. We denote by $E^{\mathbb{P}_0}[\cdot]$ the linear expectation associated with $\mathbb{P}_0$. The Itô type integrals with respect to Brownian motion are first defined for the processes X that are progressive measurable and satisfy that $E^{\mathbb{P}_0}[\int_0^T |X|^2 dt] < +\infty$. Subsequently, this definition is extended to a class of "locally integrable" processes.

Recently, Peng [73, 75] introduced a framework of time consistent nonlinear expectation $\mathbb{E}[\cdot]$, i.e., G-expectation. Under this expectation, the canonical process B is a new type of Brownian motion and it is called G-Brownian motion.

The stochastic integrals with respect to the G-Brownian motion (of the Itô type) was first discussed in Peng [73] on the space $\bar{M}_G^2([0,T])$, which is the completion of the $C_b(\Omega)$-formed simple process space $M_c^0([0,T])$ under the norm: $(\frac{1}{T}\int_0^T \bar{\mathbb{E}}[|\cdot|^2]dt)^{1/2}$. By a G-conditional expectation technique, Peng [73] showed that the Riemann-Stieltjes sum with respect to B defines an contracting and linear mapping from $M_c^0([0,T])$ to $L_G^2(\Omega_T)$, so that it can be uniquely extended to $\bar{M}_G^2([0,T])$.

In a latter literature, Peng [75] extended this definition for processes in a little larger space $M_G^2([0,T])$, which is also a completion of $M_c^0([0,T])$ but under the norm: $(\frac{1}{T}\bar{\mathbb{E}}[\int_0^T |\cdot|^2 dt])^{1/2}$.

In a more recent paper of Li and Peng [52], the G-Itô type integrals were considered on an even larger space $M_*^2([0,T])$, that is, the completion of $B_b(\Omega)$-formed simple process space $M_b^0([0,T])$ under the latter norm above. Overcoming difficulties induced by the lack of a G-conditional expectation's definition on $B_b(\Omega)$, they succeeded in proving a G-Itô inequality (instead of the Itô isometry in the classical case). Thus, the mapping defined by the Riemann-Stieltjes sum with respect to B on $M_b^0([0,T])$ is continuous and can be uniquely extended to $M_*^2([0,T])$. Li and Peng [52] observed moreover that the space $M_*^2([0,T])$ is closed under some operation, for example, the product of an arbitrary element and another bounded one in $M_*^2([0,T])$ is still in $M_*^2([0,T])$. Thanks to this advantage, an extension of Itô's integrals was given by a localization method in that paper for a class of "locally integrable" processes and this extension is well defined.

For the issue of G-Itô's formula, Peng [73] has first obtained this formula for $\Phi(X)$ when X is a G-Itô process with bounded coefficients and $\Phi \in \mathcal{C}^2(\mathbb{R}^n)$ has uniformly bounded and Lipschitz derivatives. Subsequently, Gao [27] and Zhang et al. [98] extended this result to the case that the derivatives of Φ are locally Lipschitz or uniformly continuous, respectively. Meanwhile, Peng [75] considered the case when Φ has at most polynomial growths. Based on this, an extension of G-Itô's formula is obtained in Lin [54] when X is a sum of a G-Itô process and an increasing process. Besides, Lin [53] discussed G-Tanaka's formula and studied G-Itô's formula for G-Brownian motion with a convex Φ, while Guo et al. [30] worked on the one for G-diffusion with a Φ whose second derivative allows finite number of jumps.

With the help of a creative localization method, Li and Peng [52] proved a G-Itô's formula for the G-Itô processes, in which only $\Phi \in \mathcal{C}^2(\mathbb{R}^n)$ is required, such that this formula has a parallel form with the classical one. One aim of this chapter is to consider the case similar to the one in Li and Peng [52] but a bounded variation process is added into the dynamic.

Stochastic differential equations driven by G-Brownian motion (GSDEs) are studied by many authors, such as Gao [27], Guo et al. [30], Lin [54], Lin and Bai [56] and Peng [75], mostly on the space $\bar{M}_G^p([0,T];\mathbb{R}^n)$, $p \geq 2$, and with a linear growth condition. In this chapter, the solvability of GSDEs with locally Lipschitz coefficients are studied on the space $M_w^p([0,T];\mathbb{R}^n)$ by a localization method. To the best knowledge of the authors, it is the first attempt to discuss the GSDEs without a linear growth condition.

This chapter is organized as follows: Section 2.2 introduces notation and results in the G-framework which are necessary for the following text. Section 2.3 discusses G-Itô's integrals for some "local integrable" processes. Section 2.4 introduces the G-stochastic integrals with respect to a bounded variation process and extends G-Itô's formula. Section 2.5 studies the solvability of GSDEs. Section 2.6 is the appendix that gives some complimentary proofs of G-Itô's formula.

2.2 Preliminaries

The main purpose of this section is to recall some preliminary results in the G-framework, which are necessary later in the text. The reader interested in a more detailed description of these notions is referred to Denis et al. [17], Gao [27], Li and Peng[52] and Peng [75].

Adapting to the approach in Peng [75], let Ω be the space of all $\mathbb{R}^d$-valued continuous paths with $\omega_0 = 0$ equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N}((\max_{t\in[0,N]} |\omega_t^1 - \omega_t^2|) \wedge 1),$$

B the canonical process and $C_{b,Lip}(\mathbb{R}^{d\times n})$ the collection of all bounded and Lipschitz functions on $\mathbb{R}^{d\times n}$. For a fixed $T \geq 1$, the space of finite dimensional cylinder random variables is defined by

$$L_{ip}^0(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1,\ 0 \leq t_1 \leq \dots \leq t_n \leq T,\ \varphi \in C_{b,Lip}(\mathbb{R}^{d\times n})\},$$

on which $\mathbb{E}[\cdot]$ is a sublinear functional that satisfies: for all $X, Y \in L_{ip}^0(\Omega_T)$,

(1) **Monotonicity:** if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;

(2) **Sub-additivity:** $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$;

(3) **Positive homogeneity:** $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, for all $\lambda \geq 0$;

(4) **Constant translatability:** $\mathbb{E}[X + c] = \mathbb{E}[X] + c$, for all $c \in \mathbb{R}$.

The triple $(\Omega, L_{ip}^0(\Omega_T), \mathbb{E})$ is called a sublinear expectation space.

Definition 2.1 *A d-dimensional random vector $X \in L_{ip}^0(\Omega_T)$ is G-normal distributed if for each $\varphi \in C_{b,lip}(\mathbb{R}^d)$, $u^\varphi(t,x) := \mathbb{E}[\varphi(x + \sqrt{t}X)]$ is a viscosity solution to the following PDE on $\mathbb{R}^+ \times \mathbb{R}^d$:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - G(D^2 u) = 0; \\ u|_{t=0} = \varphi, \end{cases}$$

where $G = G_X(A) : \mathbb{S}^d \to \mathbb{R}$ is defined by

$$G_X(A) := \frac{1}{2}\mathbb{E}[(AX, X)]$$

and $D^2 u = (\partial^2_{x_i x_j} u)_{i,j=1}^d$.

By Theorem 2.1 in Chapter I of Peng [75], there exists a bounded, convex and closed subset Γ of $\mathbb{R}^d$, such that for each $A \in \mathbb{S}^d$, $G_X(A)$ can be represented as

$$G_X(A) = \frac{1}{2}\sup_{\gamma\in\Gamma} \operatorname{tr}[\gamma\gamma^{\mathbf{Tr}} A].$$

We denote the d-dimensional G-normal distribution by $\mathcal{N}(0, \Sigma)$, where $\Sigma := \{\gamma\gamma^{\mathbf{Tr}} : \gamma \in \Gamma\}$.

Definition 2.2 *We call a sublinear expectation $\mathbb{E} : L_{ip}^0(\Omega_T) \to \mathbb{R}$ a G-expectation if the canonical process B is a d-dimensional G-Brownian motion under $\mathbb{E}[\cdot]$, that is, for each $0 \leq s \leq t \leq T$, the increment $B_t - B_s \sim \mathcal{N}(0, (t-s)\Sigma)$ and for all $n \in \mathbb{N}^+$, $0 \leq t_1 \leq \dots \leq t_n \leq T$ and $\varphi \in C_{b,Lip}(\mathbb{R}^{d\times n})$,*

$$\mathbb{E}[\varphi(B_{t_1}, \dots, B_{t_{n-1}}, B_{t_n} - B_{t_{n-1}})] = \mathbb{E}[\psi(B_{t_1}, \dots, B_{t_{n-1}})],$$

where $\psi(x_1, \dots, x_{n-1}) := \mathbb{E}[\varphi(x_1, \dots, x_{n-1}, \sqrt{t_n - t_{n-1}} B_1)]$.

By now, a G-expectation has been well defined on $L_{ip}^0(\Omega_T)$, one can extend its domain by following the procedures introduce in Denis et al. [17], i.e., constructing an upper expectation $\bar{\mathbb{E}}[\cdot]$:

$$\bar{\mathbb{E}}[X] := \sup_{\mathbb{P}\in\mathcal{P}_G} E^{\mathbb{P}}[X],\ X \in L^0(\Omega_T), \tag{2.1}$$

where $\mathcal{P}_G$ is a weakly compact family of probability measures that depends on Γ and $L^0(\Omega_T)$ denotes the collection of all $\mathcal{B}(\Omega_T)$ measurable random variables. This upper expectation coincides with the G-expectation $\mathbb{E}[\cdot]$ on $L^0_{ip}(\Omega_T)$ and thus, on its completion $L^1_G(\Omega_T)$ under the norm $\mathbb{E}[|\cdot|]$. Naturally, a Choquet capacity can be defined by:

$$\bar{C}(A) := \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{P}(A),\ A \in \mathcal{B}(\Omega_T),$$

and the notation of "quasi-surely" (q.s.) is introduced as follows:

Definition 2.3 *A set $A \in \mathcal{B}(\Omega_T)$ is called polar if $\bar{C}(A) = 0$. A property is said to hold quasi-surely if it holds outside a polar set. We denote by $\mathcal{N}^{\bar{C}}(\Omega_T)$ all the polar sets in $\mathcal{B}(\Omega_T)$ and we set $\bar{\mathcal{B}}(\Omega_t) := \mathcal{B}(\Omega_t) \vee \mathcal{N}^{\bar{C}}(\Omega_T)$ [1].*

For each $t \in [0, T]$, we denote by $L^0(\Omega_t)$ the collection of all $\bar{\mathcal{B}}(\Omega_t)$-measurable random variables. In view of the dual formulation of G-expectation (2.1), we can easily deduce the following result:

Theorem 2.4 (Upwards convergence theorem) *Let $\{X^n\}_{n\in\mathbb{N}} \subset L^0(\Omega_T)$ be a sequence such that $X^n \uparrow X$, q.s., and there exists a $\mathbb{P} \in \mathcal{P}_G$, $E^{\mathbb{P}}[X^0] > -\infty$, then $\bar{\mathbb{E}}[X^n] \uparrow \bar{\mathbb{E}}[X]$.*

Lemma 2.5 (Lemma 2.11 in Lin and Bai [56]) *Assume that $\{X^n\}_{n\in\mathbb{N}}$ is a sequence in $L^0(\Omega_T)$ and for a $Y \in L^0(\Omega_T)$ that satisfies $\bar{\mathbb{E}}[|Y|] < +\infty$ and all $n \in \mathbb{N}$, $X^n \geq Y$, q.s., then*

$$\bar{\mathbb{E}}[\liminf_{n\to+\infty} X^n] \leq \liminf_{n\to+\infty} \bar{\mathbb{E}}[X^n].$$

For a fixed $t \in [0, T]$, let $C_b(\Omega_t)$ denote the collection of all continuous (in ω) random variables in $L^0(\Omega_t)$ and $B_b(\Omega_t)$ the collection of all bounded random variables in $L^0(\Omega_t)$. By Theorem 52 in Denis et al. [17], for each $p \geq 1$, the completion of $L^0_{ip}(\Omega_t)$ and $C_b(\Omega_t)$ under the norm

$$||\cdot||_p := (\bar{\mathbb{E}}[|\cdot|^p])^{1/p} \tag{2.2}$$

are exactly the same, and it is denoted by $L^p_G(\Omega_t)$. In accordance with the notation in Li and Peng [52], we denote by $L^p_*(\Omega_t)$ the completion of $B_b(\Omega_t)$ under the above norm.

We recall some spaces of the integrands.

Definition 2.6 *A partition of $[0, T]$ is a finite ordered subset $\pi^N_{[0,T]} = \{t_0, t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \ldots < t_N = T$. We set*

$$\mu(\pi^N_{[0,T]}) := \max_{k=0,1,\ldots,N-1} |t_{k+1} - t_k|.$$

For each $p \geq 1$, we define

$$M^0_c([0, T]) := \left\{\eta_t = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})}(t) : \xi_k \in C_b(\Omega_{t_k})\right\}$$

and

$$M^0_b([0, T]) := \left\{\eta_t = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})}(t) : \xi_k \in B_b(\Omega_{t_k})\right\},$$

which are the spaces of simple processes.

In most literature concerning the Itô type integrals in the G-framework, the authors considered three types of spaces of integrands. For some $p \geq 1$, as the completion of $M^0_c([0, T])$ under the norm

$$\|\eta\|_{\bar{M}^p_G([0,T])} = \left(\frac{1}{T}\int_0^T \bar{\mathbb{E}}[|\eta_t|^p]dt\right)^{1/p}, \tag{2.3}$$

[1] If we append the following class of sets into $\mathcal{B}(\Omega_t)$, the argument throughout this chapter will not alter: $\mathcal{N}^{\bar{C}} := \{A \subset B : B \text{ is a } \mathbb{P}\text{−null set in } \mathcal{B}(\Omega_T),\ \text{for all } \mathbb{P} \in \mathcal{P}_G\}$.

$\bar{M}_G^p([0,T])$ was first introduced in some former literature, such as Peng [73]. Subsequently, $M_G^p([0,T])$ was defined in Peng [75] by the completion of $M_c^0([0,T])$ under the norm

$$\|\eta\|_{M_G^p([0,T])} := \left(\bar{\mathbb{E}}\left[\frac{1}{T}\int_0^T |\eta_t|^p dt\right]\right)^{1/p}. \tag{2.4}$$

By continuously extending a linear mapping, the G-Itô integrals are well defined for all the processes in these spaces.

Recently, Li and Peng [52] considered an even larger space $M_*^p([0,T])$, which is formulated by the completion of $M_b^0([0,T])$ under the norm (2.4). After proving a G-Itô's inequality (Proposition 2.11 in Li and Peng [52]), the authors defined the G-Itô type integrals for all the processes in $M_*^2([0,T])$. This extension is meaningful, so that one can define these integrals for the integrands that involve some indicator function. We introduce this definition in what follows. From now on, for each $\mathbf{a} \in \mathbb{R}^d$, $B^{\mathbf{a}}$ denotes the inner product of $\mathbf{a}$ and B.

Definition 2.7 *For each $\eta \in M_b^0([0,T])$, we define the Itô type integral*

$$\mathcal{I}_{[0,T]}(\eta) = \int_0^T \eta_t dB_t^{\mathbf{a}} := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}}^{\mathbf{a}} - B_{t_k}^{\mathbf{a}}).$$

Then, the linear mapping $\mathcal{I}_{[0,T]}$ on $M_b^0([0,T])$ can be continuously extended to $\mathcal{I}_{[0,T]} : M_*^2([0,T]) \to L_*^2(\Omega_T)$ and for each $\eta \in M_*^2([0,T])$, we define $\int_0^T \eta_t dB_t^{\mathbf{a}} := \mathcal{I}_{[0,T]}(\eta)$.

Remark 2.8 *By Proposition 2.10 in Li and Peng [52], we have the following property for the G-Itô type integral of $\eta \in M_*^2([0,T])$:*

$$\bar{\mathbb{E}}\left[\int_0^T \eta_t dB_t^{\mathbf{a}}\right] = 0, \tag{2.5}$$

Remark 2.9 *For all elements in $\bar{M}_G^p([0,T])$, the G-Itô type integrals formed by the former definition coincides with the present one, since the norm (2.3) is stronger than the one (2.4).*

Parallel to the results in the classical framework, Li and Peng [52] presented the BDG type inequality for this extension of G-Itô's integrals:

Lemma 2.10 (Proposition 3.6 in Li and Peng [52]) *Let $p \geq 2$, then for each $\eta \in M_*^p([0,T])$, the following inequality holds:*

$$\bar{\mathbb{E}}\left[\sup_{0\leq s\leq t}\left|\int_0^s \eta_u dB_u^{\mathbf{a}}\right|^p\right] \leq C_p \sigma_{\mathbf{a}\mathbf{a}^{\mathrm{Tr}}}^{p/2} \bar{\mathbb{E}}\left[\left(\int_0^t |\eta_s|^2 ds\right)^{p/2}\right], \tag{2.6}$$

where $\sigma_{\mathbf{a}\mathbf{a}^{\mathrm{Tr}}} := \bar{\mathbb{E}}[(\mathbf{a}, B_1)^2]$ and $C_p > 0$ is independent of $\mathbf{a}$, η and Γ.

Remark 2.11 *To prove this lemma, one can follow a standard argument to show in advance that the process $(\int_0^t \eta_s dB_s)_{0\leq t\leq T}$ has continuous paths q.s.. In the sequel, if not specified, we always consider a t-continuous $\bar{C}$-modification of $(\int_0^t \eta_s dB_s)_{0\leq t\leq T}$.*

Furthermore, we have the following assertion like that in the classical framework, so that we do not care the choice of the modification.

Proposition 2.12 *Suppose that X and X' are two processes that have t-continuous paths and for each $t \in [0,T]$, $X_t = X_t'$, q.s.. Then, X and X' are indistinguishable (in the q.s. sense).*

Proof: Considering all the rational points $t \in \mathbb{Q} \cap [0,T]$, we can find a polar set A, such that for all $\omega \in A^c$, $X_t(\omega) = X_t'(\omega)$, then the continuity of paths gives the desired result. □

From Definition 2.7, for a given $\eta \in M_*^p([0,T])$, $p \geq 2$, we know that for each $t \in [0,T]$, $X_t := \int_0^t \eta_s dB_s^{\mathbf{a}}$ is an element in $L_*^p(\Omega_t)$. However, if we want to consider the G-Itô type integral $\int X dB^{\mathbf{a}}$, we should first show that $X \in M_*^p([0,T])$.

Proposition 2.13 (Remark 3.12 in Li and Peng [52]) *For a given $\eta \in M_*^p([0,T])$, consider the t-continu -ous $\bar{C}$-modification X of $(\int_0^t \eta_s dB_s^{\mathbf{a}})_{0\leq t\leq T}$, then $X \in M_*^p([0,T])$.*

Remark 2.14 *This result is helpful for verifying that the iteration function Λ is a mapping from $M_*^p([0,T])$ to $M_*^p([0,T])$, when we prove the existence of a solution to some GSDE via Picard iteration.*

2.3 An extension of G-Itô integrals

In this section, we introduce the idea of Li and Peng for defining G-Itô's integrals for some "locally integrable" process. For completeness, we will repeat the whole procedures in detail. Since we shall borrow their idea to prove our main results in what follows, we will give some remarks in this section, which can be regarded as an explanation of localization methods in the G-framework. In the sequel, stopping times always refer to the ones with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ [2] that is generated by the canonical process B.

Definition 2.15 (Definition 4.4 in Li and Peng [52]) *Fixing $p \geq 1$, a stochastic process η is said to be in $M_w^p([0,T])$, if there exists a sequence of increasing stopping times $\{\sigma_m\}_{m\in\mathbb{N}}$, such that*

- *For each $m \in \mathbb{N}$, $\eta\mathbf{1}_{[0,\sigma_m]} \in M_*^p([0,T])$;*
- *$\Omega^m := \{\omega : \sigma_m(\omega) \wedge T = T\} \uparrow \bar{\Omega} \subset \Omega$, where $\bar{C}(\bar{\Omega}^c) = 0$.*

Remark 2.16 *This definition is a little different from the original one in Li and Peng [52]. In fact, the second condition here is what essentially needed for the proof of Proposition 4.7 in that paper. Taking advantage of this minor modification, the following condition in the original definition can be omitted:*

$$\int_0^T |\eta_t|^p dt < +\infty, \ q.s.. \tag{2.7}$$

We shall explain the reason. By the first condition, $\eta\mathbf{1}_{[0,\sigma_m]}$, $m \in \mathbb{N}$, are progressively measurable with respect to $\mathcal{B}([0,t]) \otimes \bar{\mathcal{B}}(\Omega_t)$ and thus, the integrals $\int_0^T |\eta_t|^p \mathbf{1}_{[0,\sigma_m]}(t)dt \in L^0(\Omega_T)$. For each $m \in \mathbb{N}$, $\eta\mathbf{1}_{[0,\sigma_m]} \in M_*^p([0,T])$ implies $\bar{\mathbb{E}}[\int_0^T |\eta_t|^p \mathbf{1}_{[0,\sigma_m]}(t)dt] < +\infty$. Therefore, we can find a sequence of polar sets $\{A^m\}_{m\in\mathbb{N}}$, such that for each $\omega \in \Omega\backslash A^m$, $\int_0^T |\eta_t(\omega)|^p \mathbf{1}_{[0,\sigma_m(\omega)]}(t)dt < +\infty$. On the other hand, for each $\omega \in \Omega^m\backslash A^m$, $\mathbf{1}_{[0,\sigma_m]}(t) \equiv 1$, then

$$\int_0^T |\eta_t(\omega)|^p dt < +\infty. \tag{2.8}$$

Letting $m \to +\infty$, we have for each $\omega \in \bar{\Omega}\backslash \cup_{m=1}^{+\infty} A^m$, (2.8) holds, which implies (2.7).

Unlike in the classical framework, only having (2.7) and defining

$$\nu_m = \inf\left\{t : \int_0^t |\eta_s|^p ds \geq m\right\} \wedge T,$$

we can not deduce that $\eta\mathbf{1}_{[0,\nu_m]} \in M_*^p([0,T])$, so that the process $\int \eta\mathbf{1}_{[0,\nu_m]}dB^{\mathbf{a}}$ may not be well defined. That is the reason why Li and Peng [52] assume the first condition.

Remark 2.17 *Suppose there is another sequence of stopping times $\{\tau_m\}_{m\in\mathbb{N}}$ that satisfies only the second condition in Definition 2.15, then $\{\tau_m \wedge \sigma_m\}_{m\in\mathbb{N}}$ satisfies also this condition. Furthermore, by Lemma 4.2 in Li and Peng [52], we know that for each $m \in \mathbb{N}$, $\eta\mathbf{1}_{[0,\tau_m\wedge\sigma_m]} \in M_*^p([0,T])$.*

For a given $\mathbf{a} \in \mathbb{R}^d$ and each $\eta \in M_*^2([0,T])$, $(\int_0^t \eta_s dB_s^{\mathbf{a}})_{0\leq t\leq T}$ has continuous paths q.s., then for a given stopping time τ, with the common sense, we can define

$$\int_0^{t\wedge\tau} \eta_s dB_s^{\mathbf{a}}(\omega)$$

by the value of the process $(\int_0^t \eta_s dB_s^{\mathbf{a}})_{0\leq t\leq T}$ at the point $(t \wedge \tau(\omega), \omega)$.

In Li and Peng [52], the definition of G-Itô's integrals for processes in $M_w^p([0,T])$, $p \geq 2$, is proved to be well posed. Their key idea is to prove the following statement: fixing $m \in \mathbb{N}$, for each $n \geq m$,

$$\mathbf{1}_{\Omega_m} \int_0^{t\wedge\tau_m} \mathbf{1}_{[0,\sigma_m]}(s)\eta_s dB_s^{\mathbf{a}} = \mathbf{1}_{\Omega_m} \int_0^{t\wedge\tau_m} \mathbf{1}_{[0,\sigma_n]}(s)\eta_s dB_s^{\mathbf{a}}.$$

Before proceeding this, we shall present a slight extension of Lemma 4.3 in Li and Peng [52]:

[2] In the literature concerning the G-framework, we always adopt the notation $\{\mathcal{B}(\Omega_t)\}_{t\geq 0}$, which coincides with $\{\mathcal{F}_t\}_{t\geq 0}$ for the case that $\Omega := C_0([0,+\infty))$.

Lemma 2.18 *For each stopping time τ and $\eta \in M_*^p([0,T])$, $p \geq 2$, we consider the t-continuous $\bar{C}$-modifications of these two processes $(\int_0^t \eta_s dB_s^{\mathbf{a}})_{0\leq t\leq T}$ and $(\int_0^t \eta_s \mathbf{1}_{[0,\tau]}(s) dB_s^{\mathbf{a}})_{0\leq t\leq T}$. Then, we can find a polar set A, such that for all $\omega \in A^c$ and $t \in [0,T]$,*

$$\int_0^{t\wedge\tau} \eta_s dB_s^{\mathbf{a}}(\omega) = \int_0^t \eta_s \mathbf{1}_{[0,\tau]}(s) dB_s^{\mathbf{a}}(\omega). \tag{2.9}$$

Proof: For each $t \in [0,T]\cap\mathbb{Q}$, by Lemma 4.3 in Li and Peng [52], we can find a polar set A^t, outside which statement (2.9) holds true. Defining $A = (\cup_{t\in[0,T]\cap\mathbb{Q}} A^t)$, by the continuity of paths on both sides, for all $\omega \in A^c$, (2.9) hold true on $[0,T]$, and A is still a polar set. □

For a given $\eta \in M_w^p([0,T])$, $p \geq 2$, with $\{\sigma_m\}_{m\in\mathbb{N}}$, we consider for each $m \in \mathbb{N}$, the t-continuous $\bar{C}$-modification of $(\int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}})_{0\leq t\leq T}$. For each m, $n \in \mathbb{N}$, $n > m$, by Lemma 2.18 we can find a polar set $\hat{A}^{m,n}$, such that for all $\omega \in (\hat{A}^{m,n})^c$, the following equality holds:

$$\begin{aligned}\int_0^{t\wedge\sigma_m} \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}}(\omega) &= \int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}}(\omega) \\ &= \int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s)\mathbf{1}_{[0,\sigma_n]}(s) dB_s^{\mathbf{a}}(\omega) \\ &= \int_0^{t\wedge\sigma_m} \eta_s \mathbf{1}_{[0,\sigma_n]}(s) dB_s^{\mathbf{a}}(\omega),\ 0\leq t\leq T.\end{aligned} \tag{2.10}$$

Define a polar set

$$\hat{A} := \bigcup_{m=1}^{+\infty} \bigcup_{n=m+1}^{+\infty} \hat{A}^{m,n}.$$

For each $m \in \mathbb{N}$ and $(\omega,t) \in \Omega\times[0,T]$, we set

$$X_t^m(\omega) := \begin{cases} \displaystyle\int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}}(\omega)\ ,\ \omega \in \hat{A}^c \cap \bar{\Omega}; \\ \qquad\qquad 0\ ,\ \text{otherwise.} \end{cases}$$

Definition 2.19 *Giving $\eta \in M_w^p([0,T])$, $p \geq 2$, for each $(\omega,t) \in \Omega\times[0,T]$, we define*

$$\int_0^t \eta_s dB_s^{\mathbf{a}}(\omega) := \lim_{m\to+\infty} X_t^m(\omega). \tag{2.11}$$

Remark 2.20 *From (2.10), for each $\omega \in \hat{A}^c$ and m, $n \in \mathbb{N}$, $n > m$, $X^n(\omega) \equiv X^m(\omega)$ on $[0,\sigma_m(\omega)\wedge T]$, which implies that this setting is consistent and that the limit in (2.11) exists pointwisely and thus, the definition above is well posed.*

On the other hand, we shall verify that only on a polar set, there will be some difference induced by choosing a sequence of stopping times different from $\{\sigma_m\}_{m\in\mathbb{N}}$. Suppose $\{\sigma'_m\}_{m\in\mathbb{N}}$ is another sequence of stopping times that satisfies the second condition in Definition 2.15 and ensures that η satisfies the first one, and suppose that X' is the process of $\int \eta dB^{\mathbf{a}}$ defined with respect to $\{\sigma'_m\}_{m\in\mathbb{N}}$. By Lemma 2.18, we can find for each $m \in \mathbb{N}$, a polar set $\tilde{A}^m$, such that for all $\omega \in (\tilde{A}^m)^c$,

$$\begin{aligned}\int_0^{t\wedge\sigma_m\wedge\sigma'_m} \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}}(\omega) &= \int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s)\mathbf{1}_{[0,\sigma'_m]}(s) dB_s^{\mathbf{a}}(\omega) \\ &= \int_0^{t\wedge\sigma_m\wedge\sigma'_m} \eta_s \mathbf{1}_{[0,\sigma'_m]}(s) dB_s^{\mathbf{a}}(\omega),\ 0\leq t\leq T,\end{aligned}$$

and we define $\tilde{A} := \cup_{m=1}^{+\infty}\tilde{A}^m$ along with a polar set $\hat{A}'$ in a similar way as $\hat{A}$ with respect to $\{\sigma_m\}_{m\in\mathbb{N}}$. For each $\omega \in (\hat{A}\cup\hat{A}'\cup\tilde{A})^c \cap\bar{\Omega}\cap\bar{\Omega}'$, we have that $X_\cdot^m(\omega) \equiv X_\cdot'^m(\omega)$ on $[0,\sigma_m(\omega)\wedge\sigma'_m(\omega)\wedge T]$. By the second condition in Definition 2.15,

$$\lim_{m\to+\infty} \bar{C}(\{\omega : \sigma_m(\omega)\wedge\sigma'_m(\omega) = T\}) = 1,$$

which implies that these two processes are indistinguishable (in the q.s. sense).

Besides, the choice of the t-continuous $\bar{C}$-modifications for $(\int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s)dB_s^{\mathbf{a}})_{0\le t\le T}$, $m\in\mathbb{N}$, will not alter the definition of $\int \eta dB$ either. This is a direct corollary of Proposition 2.12.

From the second condition in Definition 2.15, for each $\omega\in A^c$, we can find an $m\in\mathbb{N}$, such that $\sigma_m(\omega)\wedge T=T$ and thus, $\int_0^t \eta_s dB_s^{\mathbf{a}}(\omega)=X_t^m(\omega)$, $0\le t\le T$. By the continuity of paths of $\{X_t^m\}_{0\le t\le T}$, we know that paths of $(\int_0^t \eta_s dB_s^{\mathbf{a}})_{0\le t\le T}$ are t-continuous q.s..

Remark 2.21 *For a given $\eta\in M_w^p([0,T])$, $p\ge 2$, with $\{\sigma_m\}_{m\in\mathbb{N}}$, define $X_t:=\int_0^t \eta_s dB_s$, then $X\in M_w^p([0,T])$ with respect to the same sequence $\{\sigma_m\}_{m\in\mathbb{N}}$.*

2.4 Stochastic calculus with respect to a bounded variation process

In this section, we define stochastic integrals with respect to an $\mathbb{R}^n$-valued bounded variation process, and then we extend G-Itô's formula to the case where an bounded variation process appears in the dynamic. Before proceeding this, we take $\langle B\rangle$ as a special example of bounded variation process and discuss stochastic integrals with respect to it. In what follows, C denotes a positive constant whose value may vary from line to line.

2.4.1 G-stochastic integrals with respect to $\langle B\rangle$

We now consider G-stochastic integrals with respect to $\langle B^{\mathbf{a}}\rangle$, which is the quadratic variation of $B^{\mathbf{a}}$ formulated by

$$\langle B^{\mathbf{a}}\rangle_t := \lim_{\mu(\pi^N_{[0,T]})\to 0}\sum_{k=0}^{N-1}(B^{\mathbf{a}}_{t^N_{k+1}}-B^{\mathbf{a}}_{t^N_k})^2=(B^{\mathbf{a}}_t)^2-2\int_0^t B^{\mathbf{a}}_s dB^{\mathbf{a}}_s,\ \text{in } L^2_*([0,T])\ ^3. \tag{2.12}$$

We know that $\langle B^{\mathbf{a}}\rangle$ is a q.s. increasing process and by Corollary 5.7 in Chapter III of Peng [75], for each $0\le s\le t\le T$,

$$\langle B^{\mathbf{a}}\rangle_t-\langle B^{\mathbf{a}}\rangle_s\le \sigma_{\mathbf{a}\mathbf{a}^{\mathrm{Tr}}}(t-s),\ q.s.. \tag{2.13}$$

We consider the t-continuous $\bar{C}$-modification of $\langle B^{\mathbf{a}}\rangle$, and for each couple $(t,t')\in([0,T]\cap\mathbb{Q})\times([0,T]\cap\mathbb{Q})$, $t<t'$, we can find a polar set $\bar{A}^{t,t'}$, such that for all $\omega\in(\bar{A}^{t,t'})^c$, (2.13) holds. Defining a polar set

$$\bar{A}_1=\bigcup_{\substack{t,t'\in([0,T]\cap\mathbb{Q})\times([0,T]\cap\mathbb{Q})\\ t<t'}}\bar{A}^{t,t'},$$

and by the continuity of paths of $\langle B^{\mathbf{a}}\rangle$, for all $\omega\in(\bar{A}_1)^c$, (2.13) holds for all $s,t\in[0,T]$, which implies that outside $\bar{A}_1$, all paths of $\langle B^{\mathbf{a}}\rangle$ is absolutely continuous with respect to t.

Definition 2.22 *Suppose that η is an element in $M_w^p([0,T])$, $p\ge 1$, then we can find a polar set $\bar{A}_2$, such that (2.8) holds for all $\omega\in\bar{A}_2^c$. Now, we define*

$$\Big(\int_0^T \eta_t d\langle B^{\mathbf{a}}\rangle_t\Big)(\omega):=\begin{cases}\displaystyle\int_0^T \eta_t(\omega)d\langle B^{\mathbf{a}}\rangle_t(\omega)\ ,\ \omega\in(\bar{A}_1\cup\bar{A}_2)^c;\\ \qquad\qquad 0\ ,\ \text{otherwise,}\end{cases}$$

where $\int_0^t \eta_s(\omega)d\langle B\rangle_s(\omega)$ is an integral formed in the Lebesgue-Stieltjes sense.

[3] In fact, one can see by the BDG type inequality that

$$\bar{\mathbb{E}}\Big[\Big|\int_0^t\Big(\sum_{k=0}^{N-1}B_{t_k}\mathbf{1}_{[t_k,t_{k+1})}(s)-B_s\Big)dB_s\Big|^p\Big]\le\frac{2C}{p+2}\sum_{k=0}^{N-1}(t_{k+1}-t_k)^{p/2+1},$$

where the right-hand side of the inequality above is dominated by $\frac{2C}{p+2}t(\mu(\pi^N_{[0,t]}))^{p/2}$ that converges to 0. This argument implies that (2.12) can be defined in any space $L^p_*(\Omega_t)$, where $p\ge 2$.

We state that a different choice of the t-continuous $\bar{C}$-modification of $\langle B^{\mathbf{a}}\rangle$ will only alter this definition on a polar set. Because for each $\omega \in (\bar{A}_1 \cup \bar{A}_2)^c$, the Lebesgue-Stieltjes integral $\int_0^T \eta_t(\omega) d\langle B^{\mathbf{a}}\rangle_t(\omega)$ is absolutely continuous with respect to the Lebesgue measure induced by $\langle B\rangle_\cdot(\omega)$, the path $\int_0^\cdot \eta_t(\omega) d\langle B^{\mathbf{a}}\rangle_t(\omega)$ is continuous in t.

We notice that our definition for this kind of G-stochastic integrals is made path by path. Therefore, for each $\eta \in M_*^p([0,T])$, $p \geq 1$, we shall verify if this definition is compatible with the one made by continuously extending a mapping on $M_b^0([0,T])$ in Li and Peng [52]. Suppose that a sequence $\{\eta^n\}_{n\in\mathbb{N}} \subset M_b^0([0,T])$ converges to η for the norm (2.4). For each $n \in \mathbb{N}$, it is obvious that the two definitions are equivalent for the G-stochastic integral $\int \eta^n d\langle B^{\mathbf{a}}\rangle$ (indistinguishable in the q.s. sense). By a classical argument, η^n, $n \in \mathbb{N}$, is progressively measurable with respect to $\mathcal{B}([0,t]) \otimes \bar{\mathcal{B}}(\Omega_t)$ and so is η, then by the classical measure theory (cf. Revuz and Yor [80]), the random variable $\int_0^T \eta_t d\langle B\rangle_t$ defined by our definition is an element in $L^0(\Omega_T)$. Therefore, for a positive constant $C(T)$ that depends only on T, we have

$$\left|\int_0^T (\eta_t^n - \eta_t) d\langle B^{\mathbf{a}}\rangle_t\right|^p \leq \left|\int_0^T |\eta_t^n - \eta_t| d\langle B^{\mathbf{a}}\rangle_t\right|^p \leq C(T)\int_0^T |\eta_t^n - \eta_t|^p dt, \ q.s..$$

Then,

$$\begin{aligned}\bar{\mathbb{E}}\left[\left|\int_0^T (\eta_t^n - \eta_t) d\langle B^{\mathbf{a}}\rangle_t\right|^p\right] &= \sup_{\mathbb{P}\in\mathcal{P}_G} E^{\mathbb{P}}\left[\left|\int_0^T (\eta_t^n - \eta_t) d\langle B^{\mathbf{a}}\rangle_t\right|^p\right] \\ &\leq C(T) \sup_{\mathbb{P}\in\mathcal{P}_G} E^{\mathbb{P}}\left[\int_0^T |\eta_t^n - \eta_t|^p dt\right] \\ &= C(T)\bar{\mathbb{E}}\left[\int_0^T |\eta_t^n - \eta_t|^p dt\right] \to 0, \text{ as } n \to +\infty,\end{aligned} \tag{2.14}$$

which implies that these two definitions coincide on $M_*^p([0,T])$ and by our definition, the random variable $\int_0^T \eta_t d\langle B\rangle_t$ is still an element in $L_*^p(\Omega_T)$.

For two given vectors $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$, the mutual variation of $B^{\mathbf{a}}$ and $B^{\bar{\mathbf{a}}}$ is defined by $\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t := \frac{1}{4}(\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t)$. Consistently, we can define

$$\int_0^T \eta_t d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t := \frac{1}{4}\int_0^T \eta_t d\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \frac{1}{4}\int_0^T \eta_t d\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t. \tag{2.15}$$

By a standard argument, we have the following lemma, first for $\eta \in M_b^0([0,T])$, then for all $\eta \in M_*^p([0,T])$:

Lemma 2.23 *Let $p \geq 1$, $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$, $\eta \in M_*^p([0,T])$ and $0 \leq s \leq t \leq T$. Then,*

$$\bar{\mathbb{E}}\left[\sup_{0\leq s\leq t}\left|\int_0^s \eta_u d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_u\right|^p\right] \leq C_{\mathbf{a},\bar{\mathbf{a}}}^p t^{p-1}\bar{\mathbb{E}}\left[\int_0^t |\eta_s|^p ds\right], \tag{2.16}$$

where $C_{\mathbf{a},\bar{\mathbf{a}}} := \frac{1}{4}(\sigma_{(\mathbf{a}+\bar{\mathbf{a}})(\mathbf{a}+\bar{\mathbf{a}})^{\mathbf{Tr}}} + \sigma_{(\mathbf{a}-\bar{\mathbf{a}})(\mathbf{a}-\bar{\mathbf{a}})^{\mathbf{Tr}}})$.

2.4.2 Stochastic integrals with respect to a bounded variation process

Definition 2.24 *We denote by $M_{BD}([0,T];\mathbb{R}^n)$ the collection of all n-dimensional $\bar{\mathcal{B}}(\Omega_t) \otimes \mathcal{B}([0,t])$ progressively measurable processes X, whose paths $X_\cdot(\omega) : t \mapsto X_t(\omega)$ are bounded on $[0,T]$ outside a polar set A, i.e. for all $\omega \in A^c$, $|X_t(\omega)| < +\infty$, $0 \leq t \leq T$.*

Remark 2.25 *For $p \geq 1$ and $X \in M_*^p([0,T];\mathbb{R}^n)$, as we have already stated in the last section, η is progressively measurable. Thus, for each $\mathcal{O} \in \mathcal{B}(\mathbb{R})$, the process defined by $Y_\cdot := \mathbf{1}_{X_\cdot\in\mathcal{O}}$ is an element in $M_{BD}([0,T])$. We keep the notation in Lin [54]: $M_c([0,T];\mathbb{R}^n)$ denotes the collection of all processes X whose paths $X_\cdot(\omega) : t \mapsto X_t(\omega)$ are continuous in t on $[0,T]$ outside a polar set A. It is obviously that $M_c([0,T];\mathbb{R}^n)$ is a subset of $M_{BD}([0,T];\mathbb{R}^n)$.*

Definition 2.26 *We denote by $M_{FV}([0,T];\mathbb{R}^n)$ the collection of all n-dimensional processes $K \in M_c([0,T];\mathbb{R}^n)$, whose paths $K_\cdot(\omega) : t \mapsto K_t(\omega)$ are of bounded total variation over $[0,T]$ outside a polar set A, i.e. for all $\omega \in A^c$, $V_0^T(K_\cdot(\omega)) < +\infty$.*

Remark 2.27 *By a classical argument, each component K^ν, $\nu = 1,\ldots,n$, of $K \in M_{FV}([0,T];\mathbb{R}^n)$ can be viewed as the difference of two processes $K_1 - K_2$, whose components K_1^ν and K_2^ν, $\nu = 1,\ldots,n$, are q.s. t-continuous and increasing. However, for a given $K \in M_b^p([0,T];\mathbb{R}^n) \cap M_{FV}([0,T];\mathbb{R}^n)$, we have no ideal whether $V_0^T(K_\cdot(\omega)) \in M_b^p([0,T]) \cap M_{FV}([0,T])$ or not.*

In the rest part of this section, for simplicity, we use the Einstein notation, i.e. the repeat indices ν, i and j imply the summation.

Definition 2.28 *For each $X \in M_{BD}([0,T])$, we define the stochastic integral with respect to a given $K \in M_{FV}([0,T])$ by*

$$\Big(\int_0^T X_t dK_t\Big)(\omega) = \begin{cases} \displaystyle\int_0^T X_t^v(\omega) dK_t^v(\omega)\ ,\ \omega \in A^c; \\ \hfill 0\ ,\ \omega \in A, \end{cases}$$

where A is a polar set and on the complementary of which, $X_\cdot(\omega)$ is bounded and $K_\cdot(\omega)$ is of bounded total variation over $[0,T]$. The integral on the right-hand side is in the Lebesgue-Stieltjes sense.

Remark 2.29 *It is readily observed that for each $\omega \in A^c$, the Lebesgue-Stieltjes integral exists, and as we have mentioned in the last section, the random variable $\int_0^T X_t dK_t$ is an element in $L^0(\Omega_T)$. Moreover, thanks to the pathwise boundedness of X, paths of the integral $\int X dK$ are t-continuous q.s., i.e., $(\int_0^t X_s dK_s)_{0\le t\le T} \in M_c([0,T])$.*

Remark 2.30 *Letting $n = 1$, for each $X \in M_c([0,T])$ this definition is compatible with Definition 3.5 in Lin [54] that is made in the Riemann-Stieltjes sense. For these X, the stochastic integral $\int_0^T X_t dK_t$ can be q.s. approximated by the following sequence:*

$$\mathcal{V}_{[0,T]}^N(X,K)(\omega) := \sum_{k=0}^{N-1} X_{u_k}(\omega)(K_{t_{k+1}^N}(\omega) - K_{t_k^N}(\omega)),$$

where $u_k^N \in [t_k^N, t_{k+1}^N)$ By the Heine-Cantor theorem, immediately, we have

$$\mathcal{V}_{[0,T]}^N(X,K) \to \int_0^T X_t dK_t,\ q.s.,\ \text{as}\ \mu(\pi_{[0,T]}^N) \to 0.$$

Remark 2.31 *For each G-Itô process $X \in M_*^2([0,T];\mathbb{R}^n)$, we can always assume that $X \in M_c([0,T];\mathbb{R}^n)$, however, unlike in Lin [54], at this time the assertion that $\int_0^T X_t dK_t \in L_*^1(\Omega_T)$ no longer holds true, even we assume that K is a bounded element in $M_{FV}([0,T];\mathbb{R}^n) \cap M_*^2([0,T];\mathbb{R}^n)$. The reason is that we do not know the integrability of $V_0^\cdot(K)$. Only if the process K can be represented by the difference of two processes whose components are q.s. increasing processes in $M_*^2([0,T])$, we could have some similar results, but to our best knowledge, we can not have such an assertion in general.*

However, if the integrator is $\langle B\rangle$, which is also in $M_{FV}([0,T])$, then for each $X \in M_*^p([0,T])$, $\int_0^t X_s d\langle B\rangle_s \in L_*^p(\Omega_t)$, where $p \ge 1$. But for general $K \in M_{FV}([0,T])$, its paths may not have such wonderful properties, i.e., q.s. increasing and absolutely continuous with respect to t.

2.4.3 An extension of G-Itô's formula

Consider an n-dimensional process on $[0,T]$ that is a combination of a G-Itô process and a bounded variation process:

$$X_t^\nu = x_0^\nu + \int_0^t f_s^\nu ds + \int_0^t h_s^{\nu ij} d\langle B^i, B^j\rangle_s + \int_0^t g_s^{\nu j} dB_s^j + K_t^\nu,\ \nu = 1,\ldots,n, \tag{2.17}$$

where B is a d-dimensional G-Brownian motion. We first give a G-Itô's formula when Φ, $\partial_t\Phi$, $\partial_x\Phi$ and $\partial_{xx}^2\Phi$ are bounded and uniformly continuous on $[0,T] \times \mathbb{R}^n$ and K is uniformly bounded.

Lemma 2.32 *Let $0 \leq t \leq T$, $\Phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n)$ such that Φ, $\partial_t \Phi$, $\partial_x \Phi$ and $\partial^2_{xx}\Phi$ are bounded and uniformly continuous and X be given in the form of (2.17), where f^ν and $h^{\nu ij}$ are elements in $M^1_*([0,T])$, $g^{\nu j}$ is an element in $M^2_*([0,T])$, $\nu = 1, \dots, n$, i, $j = 1, \dots, d$, and K is a uniformly bounded element in $M_{FV}([0,T];\mathbb{R}^n)$ satisfying that for a positive constant α and each $0 \leq u_1 \leq T$,*

$$\lim_{u_2 \to u_1} \bar{\mathbb{E}}[|K_{u_2} - K_{u_1}|^\alpha] = 0. \tag{2.18}$$

Then,

$$\begin{aligned}\Phi(t, X_t) - \Phi(0, x_0) = &\int_0^t (\partial_t \Phi(s, X_s) + \partial_{x^\nu}\Phi(s, X_s) f^\nu_s) ds \\ &+ \int_0^t \left(\partial_{x^\nu}\Phi(s, X_s) h^{\nu ij}_s + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(s, X_s) g^{\mu i}_s g^{\nu j}_s\right) d\langle B^i, B^j\rangle_s \\ &+ \int_0^t \partial_{x^\nu}\Phi(s, X_s) g^{\nu j}_s dB^j_s + \int_0^t \partial_{x^\nu}\Phi(s, X_s) dK^\nu_s, \ q.s..\end{aligned} \tag{2.19}$$

Proof: By taking $X^0 \equiv t$, this lemma is a simple extension of Lemma 2.45 in the appendix to multidimensional case, so we omit it. □

With the help of a localization method, this formula can be generalized for a larger class of Φ and a combination of some locally integrable G-Itô process and some K that is not uniformly bounded.

Theorem 2.33 *Let $\Phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n)$ and X be given in the form of (2.17), where f^ν and $h^{\nu ij}$ are elements in $M^1_w([0,T])$, $g^{\nu j}$ is an element in $M^2_w([0,T])$, $\nu = 1, \dots, n$, i, $j = 1, \dots, d$, and K is an element in $M_{FV}([0,T];\mathbb{R}^n)$ satisfying that for a positive constant α and each $0 \leq u_1 \leq T$,*

$$\lim_{u_2 \to u_1} \bar{\mathbb{E}}[|V_0^{u_2}(K) - V_0^{u_1}(K)|^\alpha] = 0. \tag{2.20}$$

Then, (2.19) holds, in which the G-Itô type integral is defined by Definition 2.19 and the other stochastic integrals are defined in the Lebesgue-Stieltjes sense.

Proof: Without loss of generality, we prove this theorem when $n = 1$ and $d = 1$, and we assume that there exists a common sequence of increasing stopping times $\{\sigma_m\}_{m\in\mathbb{N}}$ that satisfies the second condition of Definition 2.15, such that with respect to this sequence, f, h and g satisfy the first one respectively. Define

$$\mu_m := \inf\{t : |X_t| + |K_0| + V_0^t(K) \geq m\} \wedge T \text{ and } \tau_m := \mu_m \wedge \sigma_m.$$

From the argument at the end of Section 2.3 and in Subsection 2.4.1, we know that paths of the G-Itô part of X (we denote it by M^X like in Lin [54]) are q.s. t-continuous and thus, uniformly continuous and bounded on $[0,T]$. From this fact along with that paths of K are q.s. t-continuous and of bounded variation over $[0,T]$, $\{\mu_m\}_{m\in\mathbb{N}}$ satisfies the second condition in Definition 2.15, i.e.,

$$\Omega^m := \{\omega : \mu_m(\omega) \wedge T = T\} \uparrow \bar{\Omega} \subset \Omega, \text{ where } \bar{C}(\bar{\Omega}^c) = 0.$$

Thus, by Remark 2.17, $\{\tau_m\}_{m\in\mathbb{N}}$ is a suitable sequence to replace $\{\sigma_m\}_{m\in\mathbb{N}}$, such that with respect to this new sequence f, h and g satisfy the first condition in Definition 2.15 respectively, which implies $f\mathbf{1}_{[0,\tau_m]} \in M^1_*([0,T])$, $h\mathbf{1}_{[0,\tau_m]} \in M^1_*([0,T])$ and $g\mathbf{1}_{[0,\tau_m]} \in M^2_*([0,T])$. By the definition of G-stochastic integrals for local integrable processes, for each $m \in \mathbb{N}$, $X_{t\wedge\tau_m}$ can be written in the following form:

$$X_{t\wedge\tau_m} = x + \int_0^t f_s \mathbf{1}_{[0,\tau_m]}(s) ds + \int_0^t h_s \mathbf{1}_{[0,\tau_m]}(s) d\langle B\rangle_s + \int_0^t g_s \mathbf{1}_{[0,\tau_m]}(s) dB_s + K_{t\wedge\tau_m}, \ q.s..$$

As $V_0^\cdot(K)$ is an increasing process, for each $m \in \mathbb{N}$, $V_0^{\cdot\wedge\tau_m}(K)$ still satisfies (2.20). Then, from the equality that $V_0^{t\wedge\tau_m}(K) \equiv V_0^t(K_{\cdot\wedge\tau_m})$ on $[0,T]$, we have, for each $0 \leq u_1 \leq u_2 \leq T$,

$$\begin{aligned}\bar{\mathbb{E}}[|K_{u_2\wedge\tau_m} - K_{u_1\wedge\tau_m}|^\alpha] &\leq \bar{\mathbb{E}}[|V_0^{u_2}(K_{\cdot\wedge\tau_m}) - V_0^{u_1}(K_{\cdot\wedge\tau_m})|^\alpha] \\ &= \bar{\mathbb{E}}[|V_0^{u_2\wedge\tau_m}(K) - V_0^{u_1\wedge\tau_m}(K)|^\alpha] \to 0, \text{ as } u_2 \to u_1.\end{aligned} \tag{2.21}$$

On the other hand, we can see that K is uniformly bounded by m, then following the argument in Remark 2.43, we know that $K_{\cdot\wedge\tau_m} \in M_*^p([0,T])$, for any $p \geq 1$, and thus, $X_{\cdot\wedge\tau_m} \in M_*^2([0,T])$.

Since $\Phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)$ such that Φ, $\frac{d\Phi}{dt}$, $\frac{d\Phi}{dx}$ and $\frac{d^2\Phi}{dx^2}$ are bounded and uniformly continuous on the compact set $\{x : |x| \leq m\}$. Therefore, we can apply Lemma 2.32 to $\Phi(t, X_{t\wedge\tau})$ and obtain

$$\begin{aligned}\Phi(t, X_{t\wedge\tau_k}) - \Phi(0, x_0) &= \int_0^t \left(\frac{d\Phi}{dt}(s, X_s) + \frac{d\Phi}{dx}(s, X_s) f_s\right)\mathbf{1}_{[0,\tau_m]}(s)ds \\ &+ \int_0^t \left(\frac{d\Phi}{dx}(s, X_s) h_s + \frac{1}{2}\frac{d\Phi^2}{dx^2}(s, X_s) g_s^2\right)\mathbf{1}_{[0,\tau_m]}(s)d\langle B\rangle_s \\ &+ \int_0^t \frac{d\Phi}{dx}(s, X_s) g_s \mathbf{1}_{[0,\tau_m]}(s)dB_s + \int_0^t \frac{d\Phi}{dx}(s, X_s)\mathbf{1}_{[0,\tau_m]}(s)dK_{s\wedge\tau_m},\ q.s..\end{aligned} \tag{2.22}$$

We take the term $\int_0^t \frac{d\Phi}{\partial x}(s, X_s) g_s \mathbf{1}_{[0,\tau_m]}(s)dB_s$ as an example to explain how we can pass to limit for the G-Itô part as $m \to +\infty$. Since $\frac{d^2\Phi}{dx^2}$ is bounded on $\{x : |x| \leq m\}$, one can see that $\frac{d\Phi}{dx}$ is uniformly Lipschitz in x on this compact set. Then, from the fact that $X_{\cdot\wedge\tau_m} \in M_*^2([0,T])$, we deduce that $\frac{d\Phi}{dx}(\cdot, X_{\cdot\wedge\tau_m}) \in M_*^2([0,T])$ (cf. the proof of Lemma 2.36). Furthermore, by Lemma 4.2 in Li and Peng [52], we have $\frac{d\Phi}{dx}(\cdot, X_\cdot)\mathbf{1}_{[0,\tau_m]}(\cdot) = \frac{d\Phi}{dx}(\cdot, X_{\cdot\wedge\tau_m})\mathbf{1}_{[0,\tau_m]}(\cdot) \in M_*^2([0,T])$. Finally, because $\frac{d\Phi}{dx}$ is also bounded on $\{x : |x| \leq m\}$, by Proposition 3.11 in Li and Peng [52], $\frac{d\Phi}{dx}(\cdot, X_\cdot)g_\cdot\mathbf{1}_{[0,\tau_m]}(\cdot) \in M_*^2([0,T])$. Therefore, $\frac{d\Phi}{dx}(\cdot, X_\cdot)g_\cdot$ well satisfies the first condition in Definition 2.15 with respect to $\{\tau_m\}_{m\in\mathbb{N}}$ and thus, it is an element in $M_w^2([0,T])$ and this term can be well defined q.s., as $m \to +\infty$.

Now we consider the last term in (2.22). For each $\omega \in \bar{\Omega}$ ($\bar{\Omega}$ is corresponding to the new sequence $\{\mu_m\}_{m\in\mathbb{N}}$), there exists an $m \in \mathbb{N}$, such that $\omega \in \Omega^m$ and $\tau_m(\omega) \wedge T = T$. Hence, for all $l \in \mathbb{N}$, $l \geq m$,

$$\int_0^t \frac{\partial\Phi}{\partial x}(s, X_s(\omega))\mathbf{1}_{[0,\tau_l(\omega)]}(s)dK_{s\wedge\tau_l(\omega)}(\omega) = \int_0^t \frac{\partial\Phi}{\partial x}(s, X_s(\omega))\mathbf{1}_{[0,\tau_m(\omega)]}(s)dK_{s\wedge\tau_m(\omega)}(\omega).$$

Moreover, paths of X are t-continuous outside a polar set A, then for all $\omega \in \bar{\Omega}\backslash A$, the integrals on both sides of the equality above exist and

$$\int_0^t \frac{\partial\Phi}{\partial x}(s, X_s(\omega))(s)dK_s(\omega) := \lim_{m\to+\infty}\int_0^t \frac{\partial\Phi}{\partial x}(s, X_s(\omega))\mathbf{1}_{[0,\tau_m(\omega)]}(s)dK_{s\wedge\tau_m(\omega)}(\omega).$$

We complete the proof. □

Remark 2.34 *In this theorem, in order to ensure that (2.21) holds true, we assume that $V_0^\cdot(K)$ satisfies (2.20), which is stronger than the corresponding condition in Lemma 2.32. For $n = 1$, if K is an increasing process, then this condition coincides with that K satisfies (2.18).*

2.5 G-stochastic differential equations

In this section, we consider an n-dimensional GSDE of the following form:

$$X_t = x + \int_0^t f(s, X_s)ds + \int_0^t h(s, X_s)d\langle B, B\rangle_s + \int_0^t g(s, X_s)dB_s,\ 0 \leq t \leq T,\ q.s., \tag{2.23}$$

where $x \in \mathbb{R}^n$ is the initial value, B is the d-dimensional Brownian motion, $\langle B, B\rangle = (\langle B^i, B^j\rangle)_{i,j=1,\dots,d}$ is the mutual variation matrix of B and f, h and g are functions such that for fixed $\omega \in \Omega$, $f(\cdot,\cdot)(\omega) = (f^1(\cdot,\cdot)(\omega), \dots, f^n(\cdot,\cdot)(\omega))^{\mathbf{Tr}} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $h(\cdot,\cdot)(\omega) = (h_{ij}^\nu(\cdot,\cdot)(\omega))_{i,j=1,\dots,d}^{\nu=1,\dots,n} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times d^2}$ and $g(\cdot,\cdot)(\omega) = (g_j^\nu(\cdot,\cdot)(\omega))_{j=1,\dots,d}^{\nu=1,\dots,n} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times d}$.

We introduce the following conditions:

(H1) For some $p \geq 2$ and each $x \in \mathbb{R}$, $f^\nu(\cdot, x)$, $h_{ij}^\nu(\cdot, x)$, $g_j^\nu(\cdot, x) \in M_*^p([0,T])$, $\nu = 1, \dots, n$, $i, j = 1, \dots, d$;

(H2) The coefficients f, h and g are uniformly Lipschitz in x, i.e., for each $t \in [0,T]$ and $x, x' \in \mathbb{R}^n$,

$$|f(t,x) - f(t,x')| + ||h(t,x) - h(t,x')|| + ||g(t,x) - g(t,x')|| \leq C_L|x - x'|,\ q.s.,$$

where $||\cdot||$ is the Hilbert-Schmidt norm of a matrix. Here, q.s. means this inequality holds for all the ω outside a polar set A independent of t.

We notice that the domain of coefficients here is larger than the ones in other articles that we mentioned in the section of introduction.

Theorem 2.35 *Let (H1) and (H2) hold. Then, there exists a unique process $X \in M_*^p([0,T];\mathbb{R}^n)$ that has t-continuous satisfying (2.23). For two initial values x, $y \in \mathbb{R}^n$, let X^x and X^y be two solutions of (2.23) respectively with the initial values x and y, then there exists a constant $C > 0$ that depends only on p, n, T and C_L, such that*

$$\bar{E}[\sup_{t\in[0,T]} |X_t^x - X_t^y|^p] \leq C|x - y|^p. \tag{2.24}$$

We shall prove the existence part by the Picard iteration. Before proceeding this, we should prove the following lemma, which ensures that all stochastic integrals are well defined in each iterative step. In the sequel, a constant $C > 0$ that depends only on p, n, T and C_L may vary from line to line.

Lemma 2.36 *For some $p \geq 1$, ζ is a function that satisfies $\zeta(\cdot,x) \in M_*^p([0,T])$ for each $x \in \mathbb{R}^n$. We assume moreover that $\zeta(\cdot,x)$ satisfies a uniform Lipschitz condition, i.e., for each $t \in [0,T]$ and each x_1, $x_2 \in \mathbb{R}^n$, $|\zeta(t,x_1) - \zeta(t,x_2)| \leq C_L|x_1 - x_2|$. Then, for each $X \in M_*^p([0,T];\mathbb{R}^n)$, $\zeta(\cdot,X.)$ is an element in $M_*^p([0,T])$.*

Proof: This proof is similar to the one of Lemma 5.1 in Lin and Bai [56]. Without loss of generality, we only give the proof to the one dimensional case. Suppose that X can be approximated by a sequence $\{X^N\}_{N\in\mathbb{N}} \subset M_b^0([0,T])$ of the form below:

$$X_t^N := \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})}(t),$$

where $\xi_k \in B_b(\Omega_{t_k})$, then

$$\bar{\mathbb{E}}\left[\int_0^T |\zeta(t,X_t^N) - \zeta(t,X_t)|^p dt\right] \leq C_L\bar{\mathbb{E}}\left[\int_0^T |X_t^N - X_t|^p dt\right] \to 0, \text{ as } N \to +\infty.$$

To obtain the desired result, we only need to prove that for each $k \in \mathbb{N}$, $\zeta(\cdot,\xi_k)\mathbf{1}_{[t_k,t_{k+1})}(\cdot) \in M_*^p([0,T])$. Since $\xi_k \in B_b(\Omega_{t_k})$, there exists an $M > 0$, such that $\xi_k \in [-M,M]$, which is a compact subset in $\mathbb{R}$. For each $n \in \mathbb{N}$, we can find an open cover $\{G_i\}_{i\in I}$ of $\mathbb{R}$, such that $\lambda(G_i) < \frac{1}{n}$, $i \in I$. By the partition of unity theorem, there exists a family of $C_0^\infty(\mathbb{R})$ function $\{\phi_i^n\}_{i\in I}$, such that for each $i \in I$, $\text{supp}(\phi_i^n) \subset G_i$, $0 \leq \phi_i^n \leq 1$, and for each $x \in \mathbb{R}$, $\sum_{i\in I} \phi_i^n(x) = 1$. Moreover, there exists a finite number of ϕ_i^n, such that for each $x \in [-M,M]$,

$$\sum_{i=1}^{N(n)} \phi_i^n(x) = 1.$$

Choosing for each $i = 1,\ldots,N(n)$ a point x_i^n such that $\phi_i^n(x_i^n) > 0$, we set

$$\zeta^n(t,x) = \sum_{i=1}^{N(n)} \zeta(t,x_i^n)\phi_i^n(x).$$

Then,

$$\begin{aligned} &|\zeta^n(t,\xi_k)\mathbf{1}_{[t_k,t_k+1)}(t) - \zeta(t,\xi_k)\mathbf{1}_{[t_k,t_k+1)}(t)| \\ &\quad\leq \sum_{i=1}^{N(n)} |\zeta(t,\xi_k) - \zeta(t,x_i^n)|\phi_i^n(\xi_k) \leq \frac{C_L}{n},\ t_k \leq t < t_{k+1}, \end{aligned} \tag{2.25}$$

which implies that $\zeta^n(\cdot,\xi_k)\mathbf{1}_{[t_k,t_k+1)}(\cdot)$ converges to $\zeta(\cdot,\xi_k)\mathbf{1}_{[t_k,t_{k+1})}(\cdot)$ under the norm (2.4).

On the other hand, applying Proposition 3.11 in Li and Peng [52], we have $\zeta(t,x_i^n)\phi_i^n(\xi_k)\mathbf{1}_{[t_k,t_k+1)} \in M_*^p([0,T])$, for $i=1,\dots,N(n)$, which implies for each $n\in\mathbb{N}$, $\zeta^n(\cdot,\xi_k)\mathbf{1}_{[t_k,t_k+1)}(\cdot) \in M_*^p([0,T])$. Then, (2.25) yields that $\zeta(\cdot,\xi_k)\mathbf{1}_{[t_k,t_k+1)}(\cdot)\in M_*^p([0,T])$. □

Proof to Theorem 2.35: We start with the proof of the existence. We set $X^0\equiv x$ and define a Picard sequence $\{X^m\}_{m\in\mathbb{N}}$ in the following way:

$$X_t^{m+1}=x+\int_0^t f(s,X_s^m)ds+\int_0^t h(s,X_s^m)d\langle B,B\rangle_s+\int_0^t g(s,X_s^m)dB_s,\ 0\le t\le T,\ m\ge 1.$$

By Lemma 2.36 and the conclusion in previous sections, we know that this Picard sequence is well defined and that for each $m\in\mathbb{N}$, X^m is an element in $M_*^p([0,T])$ that has t-continuous paths.

First, we establish an a priori estimate uniform in m for $\{\bar{\mathbb{E}}[\sup\limits_{0\le t\le T}|X_t^m|^p]\}_{m\in\mathbb{N}}$. For each $m\in\mathbb{N}$, we have

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_t^{m+1}|^p]\le C\bigg(|x|^p+\bar{\mathbb{E}}\bigg[\int_0^T|f(t,0)|^pdt\bigg]+\bar{\mathbb{E}}\bigg[\int_0^T||h(t,0)||^pdt\bigg]\\ +\bar{\mathbb{E}}\bigg[\int_0^T||g(t,0)||^pdt\bigg]+\int_0^t\bar{\mathbb{E}}[\sup_{0\le u\le s}|X_u^m|^p]ds\bigg).\end{aligned}$$

Because $f(\cdot,0)$, $h_{ij}(\cdot,0)$, $g_j(\cdot,0)\in M_*^p([0,T];\mathbb{R}^n)$, $i,j=1,\dots,d$, we know that

$$\bar{\mathbb{E}}\bigg[\int_0^T|f(t,0)|^pdt\bigg]+\bar{\mathbb{E}}\bigg[\int_0^T||h(t,0)||^pdt\bigg]+\bar{\mathbb{E}}\bigg[\int_0^T||g(t,0)||^pdt\bigg]<+\infty.$$

By recurrence, it is easy to verify that for all $m\in\mathbb{N}$,

$$\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s^m|^p]\le p(t),\ 0\le t\le T,$$

where $p(\cdot)$ is the solution to the following ordinary differential equation:

$$p(t)=C\bigg(|x|^p+\bar{\mathbb{E}}\bigg[\int_0^T|f(t,0)|^pdt\bigg]+\bar{\mathbb{E}}\bigg[\int_0^T||h(t,0)||^pdt\bigg]+\bar{\mathbb{E}}\bigg[\int_0^T||g(t,0)||^pdt\bigg]+\int_0^t p(s)ds\bigg),$$

and $p(\cdot)$ is continuous and thus, bounded on $[0,T]$.

Secondly, for each l, $m\in\mathbb{N}$, we define

$$u_t^{m+1,l}:=\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s^{m+l+1}-X_s^{m+1}|^p],\ 0\le t\le T.$$

By Lemma 2.10 and 2.23, we obtain

$$u_t^{m+1,l}\le C\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s^{m+l}-X_s^m|^p]=C\int_0^t u_s^{m,l}ds. \tag{2.26}$$

Set

$$v_t^m:=\sup_{l\in\mathbb{N}}u_t^{m,l},\ 0\le t\le T,$$

then

$$0\le u_t^{m+1,l}\le C\sup_{l\in\mathbb{N}}\int_0^t u_s^{m,l}ds\le C\int_0^t\sup_{l\in\mathbb{N}}u_s^{m,l}ds=C\int_0^t v_s^m ds.$$

Taking the supremum over all $l\in\mathbb{N}$ on the left-hand side, we obtain

$$0\le v_t^{m+1}=\sup_{l\in\mathbb{N}}u_t^{m+1,l}\le C\int_0^t v_s^m ds.$$

Finally, we define
$$\alpha_t := \limsup_{m\to+\infty} v_t^m,\ 0\le t\le T.$$
It is easy to find that $v_t^m \le C_1 p(t)$, where $C_1>0$ is independent of m. By the Fatou-Lebesgue theorem, we have
$$0\le \alpha_t \le C\int_0^t \alpha_s ds.$$
Gronwall's lemma gives
$$\alpha_t = 0,\ 0\le t\le T,$$
which implies that $\{X^m\}_{m\in\mathbb{N}}$ is a Cauchy sequence under the norm
$$(\mathbb{E}[\sup_{0\le t\le T}|\cdot|^p])^{\frac{1}{p}}. \tag{2.27}$$
Following a standard argument, we can construct a process $X\in M_*^p([0,T])$ that has t-continuous paths and is the limit of $\{X^m\}_{m\in\mathbb{N}}$ under the norm (2.27). On the other hand, we have
$$\begin{aligned}
&\bar{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t (f(s,X_s^m)-f(s,X_s))ds\right|^p\right] \le C\int_0^T \bar{\mathbb{E}}[|X_t^m - X_t|^p]dt;\\
&\bar{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t (h(s,X_s^m)-h(s,X_s))d\langle B,B\rangle_s\right|^p\right] \le C\int_0^T \bar{\mathbb{E}}[|X_t^m - X_t|^p]dt;\\
&\bar{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t (g(s,X_s^m)-g(s,X_s))dB_s\right|^p\right] \le C\int_0^T \bar{\mathbb{E}}[|X_t^m - X_t|^p]dt,
\end{aligned}$$
which converge to 0 as a result of $\bar{\mathbb{E}}[\sup_{0\le t\le T}|X_t^m - X_t|^p]\to 0$. Now, we can conclude that X is a solution of (2.23).

For two initial values x, $y\in\mathbb{R}^n$, we calculate in a similar way to (2.26), then we obtain
$$\bar{\mathbb{E}}[\sup_{s\in[0,t]}|X_s^x - X_s^y|^p] \le C\bigg(|x-y|^p + \int_0^t \bar{\mathbb{E}}[\sup_{u\in[0,s]}|X_u^x - X_u^y|^p]ds\bigg),$$
which implies (2.24) by Gronwall's Lemma. From this dependence of solutions with respect to initial values, we obtain the uniqueness of the solution to (2.23) in $M_*^p([0,T];\mathbb{R}^n)$. □

With a localization method, we shall discuss two kinds of GSDEs, both of them have coefficients that are locally Lipschitz in x, where in the first one, the Lipschitz constant varies in term of t; in the second one, the Lipschitz constant varies in term of x. Here below are the conditions for the first case:

(H1') For some $p\ge 2$ and each $x\in\mathbb{R}$, $f^\nu(\cdot,x)$, $h_{ij}^\nu(\cdot,x)$, $g_j^\nu(\cdot,x)\in M_w^p([0,T])$ with respect to a common sequence of stopping times $\{\sigma_m\}_{m\in\mathbb{N}}$, $\nu=1,\dots,n$, $i,j=1,\dots,d$;

(H2') Outside a polar set A, the coefficients f, h and g are locally Lipschitz in x, i.e., for each $t\in[0,T]$ and x, $x'\in\mathbb{R}^n$,
$$|f(t,x)(\omega)-f(t,x')(\omega)| + ||h(t,x)(\omega)-h(t,x')(\omega)|| + ||g(t,x)(\omega)-g(t,x')(\omega)|| \le C_t(\omega)|x-x'|,$$
where C is a positive process whose paths $C_\cdot(\omega)$ are continuous on $[0,T]$ outside the polar set A.

Theorem 2.37 *Let (H1') and (H2') hold. Then, there exists a unique process $X\in M_w^p([0,T];\mathbb{R}^n)$ that has t-continuous paths on $[0,T]$ satisfying (2.23).*

Proof: For each $m\in\mathbb{N}$, we define
$$\mu_m := \inf\{t: C_t\ge m\}\wedge T \text{ and } \tau_m := \mu_m\wedge\sigma_m.$$
Since $C_\cdot(\omega)$ is t-continuous outside a polar set A, it is easily observed that $\{\tau_m\}_{m\in\mathbb{N}}$ is a suitable sequence such that with respect to this sequence, for each $x\in\mathbb{R}^n$, the coefficients $f^\nu(\cdot,x)$, $h_{ij}^\nu(\cdot,x)$ and $g_j^\nu(\cdot,x)$, $\nu=1,\dots,n$, $i,j=1,\dots,d$, satisfy the first condition of Definition 2.15. On the other hand, for each

m, $f\mathbf{1}_{[0,\tau_m]}$, $h\mathbf{1}_{[0,\tau_m]}$ and $g\mathbf{1}_{[0,\tau_m]}$ are uniformly Lipschitz with the constant m, thus, the GSDE (2.23) with the coefficients $f\mathbf{1}_{[0,\tau_m]}$, $h\mathbf{1}_{[0,\tau_m]}$ and $g\mathbf{1}_{[0,\tau_m]}$ admits a unique solution in $M_*^p([0,T];\mathbb{R}^n)$, which is denoted by X^m. From the proof of the last theorem, we also know that for $k \geq m$, $f^\nu(\cdot, X_\cdot^k)\mathbf{1}_{[0,\tau_m]}$, $h_{ij}^\nu(\cdot, X_\cdot^k)\mathbf{1}_{[0,\tau_m]}$, $g_j^\nu(\cdot, X_\cdot^k)\mathbf{1}_{[0,\tau_m]} \in M_*^p([0,T])$. In what follows, we will verify that, this sequence of solutions are consistent, i.e., for each $m \in \mathbb{N}$, the solution $X_\cdot^m$ and $X_\cdot^{m+1}$ are indistinguishable (in the q.s. sense) on $[0,\tau_m]$. For each $m \in \mathbb{N}$, we calculate

$$\begin{aligned}
\bar{\mathbb{E}}[\sup_{0\leq s\leq t} |X_{s\wedge\tau_m}^{m+1} - X_{s\wedge\tau_m}^m|^p] &= \bar{\mathbb{E}}[\sup_{0\leq s\leq t} (|X_s^{m+1} - X_s^m|^p \mathbf{1}_{[0,\tau_m]}(s))] \\
&= \bar{\mathbb{E}}\bigg[\sup_{0\leq s\leq t}\bigg(\bigg|\int_0^s (f(u, X_u^{m+1})\mathbf{1}_{[0,\tau_{m+1}]}(u) - f(u, X_u^m)\mathbf{1}_{[0,\tau_m]}(u))du \\
&+ \int_0^s (h(u, X_u^{m+1})\mathbf{1}_{[0,\tau_{m+1}]}(u) - h(u, X_u^m)\mathbf{1}_{[0,\tau_m]}(u))d\langle B,B\rangle_u \\
&+ \int_0^s (g(u, X_u^{m+1})\mathbf{1}_{[0,\tau_{m+1}]}(u) - g(u, X_u^m)\mathbf{1}_{[0,\tau_m]}(u))dB_u\bigg|^p \mathbf{1}_{[0,\tau_m]}(s)\bigg)\bigg] \\
&= \bar{\mathbb{E}}\bigg[\sup_{0\leq s\leq t}\bigg(\bigg|\int_0^s (f(u, X_u^{m+1}) - f(u, X_u^m))\mathbf{1}_{[0,\tau_m]}(u)du \\
&+ \int_0^s (h(u, X_u^{m+1}) - h(u, X_u^m))\mathbf{1}_{[0,\tau_m]}(u)d\langle B,B\rangle_u \\
&+ \int_0^s (g(u, X_u^{m+1}) - g(u, X_u^m))\mathbf{1}_{[0,\tau_m]}(u)dB_u\bigg|^p\bigg)\bigg] \\
&\leq C_p m^p \bar{\mathbb{E}}\bigg[\int_0^t |X_s^{m+1} - X_s^m|^p \mathbf{1}_{[0,\tau_m]}(s)ds\bigg] \\
&\leq C_p m^p \int_0^t \bar{\mathbb{E}}[\sup_{0\leq u\leq s} (|X_u^{m+1} - X_u^m|^p \mathbf{1}_{[0,\tau_m]}(u))]ds \\
&\leq C_p m^p \int_0^t \bar{\mathbb{E}}[\sup_{0\leq u\leq s} |X_{u\wedge\tau_m}^{m+1} - X_{u\wedge\tau_m}^m|^p]ds,\ 0 \leq t \leq T.
\end{aligned}$$

Gronwall's lemma implies that $\bar{\mathbb{E}}[\sup_{0\leq t\leq T} |X_{t\wedge\tau_m}^{m+1} - X_{t\wedge\tau_m}^m|^p] = 0$. Therefore, we can find a polar set $\bar{A}$, such that for all $\omega \in \bar{A}^c$ and each $m \in \mathbb{N}$, the path $X_\cdot^m(\omega)$ are t-continuous, and that $X_{\cdot\wedge\tau_m}^m(\omega) \equiv X_{\cdot\wedge\tau_m}^{m+1}$. For all $\omega \in \bar{\Omega}\backslash(A \cup \bar{A})$, there exists an $m \in \mathbb{N}$, such that for all $k \geq m$, $\tau_k(\omega) = T$. Then, we define

$$X_t(\omega) := \begin{cases} \lim_{m\to+\infty} X_t^m(\omega)\ ,\ \omega \in \bar{\Omega}\backslash(A \cup \bar{A}); \\ \qquad 0\ ,\ \text{otherwise}. \end{cases}$$

One can easily see that $X \in M_w^p([0,T];\mathbb{R}^n)$ with respect to the sequence $\{\tau_m\}_{m\in\mathbb{N}}$ and has t-continuous paths, then stochastic integrals in GSDE (2.23) can be well defined and X is a solution of this equation.

Suppose there are two processes X and $X' \in M_w^p([0,T])$ with two sequences $\{\mu_m\}_{m\in\mathbb{N}}$ and $\{\mu'_m\}_{m\in\mathbb{N}}$ such that both of them satisfy GSDE (2.23). We define a new sequence by $\tau_m := \sigma_m \wedge \mu_m \wedge \mu'_m$, for $m \in \mathbb{N}$, then we can verify that, $f^\nu(\cdot, X_\cdot)$, $f^\nu(\cdot, X'_\cdot)$, $h_{ij}^\nu(\cdot, X_\cdot)$, $h_{ij}^\nu(\cdot, X'_\cdot)$, $g_j^\nu(\cdot, X_\cdot)$, $g_j^\nu(\cdot, X'_\cdot) \in M_w^p([0,T])$. Subsequently, we apply G-Itô's formula (cf. Theorem 5.4 in Li and Peng [52]) to $|X_t - X'_t|^2$, then multiple $\mathbf{1}_{[0,\tau_m]}$ on both sides. One can readily obtain that for each $m \in \mathbb{N}$, $X_\cdot$ and $X'_\cdot$ are indistinguishable (in the q.s. sense) on $[0,\tau_m]$, from which we have the uniqueness. □

In what follows, we consider a time-homogeneous GSDE, whose coefficients satisfy both a local Lipschitz condition and a Lyapunov condition. Detailed discussion for this kind of SDEs in the classical framework can be found in Has'minskiǐ [32] or Briand [6]. We will consider the following two assumptions:

(H2") The coefficients f^ν, h_{ij}^ν, $g_j^\nu : \mathbb{R}^n \to \mathbb{R}$ are deterministic functions, $\nu = 1,\ldots,n$, $i,j = 1,\ldots,d$, such that f, h and g are locally Lipschitz in x, i.e., for each x, $x' \in \{x : |x| \leq R\}$, there exists a positive constant C_R that only depends on R, such that

$$|f(x) - f(x')| + ||h(x) - h(x')|| + ||g(x) - g(x')|| \leq C_R|x - x'|;$$

(H3") There exists a deterministic Lyapunov function $V \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n)$ satisfying $V \geq 1$, such that

$$\inf_{|x| \geq R} \inf_{t \in [0,T]} V(t,x) \to +\infty, \text{ as } R \to +\infty,$$

and there exists a constant $C_{LY} \geq 0$, such that for all $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\mathcal{L}V(t,x) \leq C_{LY} V(t,x),$$

where $\mathcal{L}$ is a differential operator defined by

$$\mathcal{L}V(t,x) := \partial_t V(t,x) + \partial_{x^\nu} V(t,x) f^\nu(x) + \sup_{\mathcal{S} \in \Sigma} \left(\partial_{x^\nu} V(t,x) h_{ij}^\nu(x) \sigma_{ij}^{\mathcal{S}} + \frac{1}{2} \partial^2_{x^\mu x^\nu} V(t,x) g_i^\mu(x) g_j^\nu(x) \sigma_{ij}^{\mathcal{S}} \right),$$

in which $\mathcal{S} := (\sigma_{ij}^{\mathcal{S}})_{i,j=1}^d \in \Sigma \subset \mathbb{S}^d$.

Theorem 2.38 *Let (H2") and (H3") hold. Then, there exists a unique process $X \in M_w^p([0,T];\mathbb{R}^n)$ to GSDE (2.23) that has t-continuous paths on $[0,T]$, and the following estimate holds:*

$$\bar{\mathbb{E}}[V(t, X_t^x)] \leq e^{C_{LY} T} V(0,x).$$

Proof: For each $N \in \mathbb{N}$, we first consider the following truncated GSDE:

$$X_t^N = x + \int_0^t f^N(X_s^N) ds + \int_0^t h^N(X_s^N) d\langle B, B \rangle_s + \int_0^t g^N(X_s^N) dB_s, \ 0 \leq t \leq T, \ q.s., \tag{2.28}$$

where $(b^\nu)^N$, $(h_{ij}^\nu)^N$ and $(g_j^\nu)^N$, $\nu = 1, \ldots, n$, $i, j = 1, \ldots, d$, are defined in the following form:

$$\zeta^N(x) = \begin{cases} \zeta(x) & , \text{if } |x| \leq N; \\ \zeta(Nx/|x|), & \text{if } |x| > N. \end{cases}$$

It is easy to verify that b^N, h^N and g^N are all bounded functions and uniformly Lipschitz in x and thus, (H1) and (H2) are both satisfied. Then, by Theorem 2.35, (2.28) admits a unique solution X^N in $M_*^p([0,T];\mathbb{R}^n)$, which has t-continuous paths outside a polar set.

Define a sequence of stopping times by

$$\tau_N := \inf\{t : |X_t^N| \geq N\} \wedge T.$$

Thanks to Lemma 2.18, we can deduce from (2.28) that

$$\begin{aligned} X_{t \wedge \tau_N}^N &= x + \int_0^t f^N(X_s^N) \mathbf{1}_{[0,\tau_N]}(s) ds \\ &+ \int_0^t h^N(X_s^N) \mathbf{1}_{[0,\tau_N]}(s) d\langle B, B \rangle_s + \int_0^t g^N(X_s^N) \mathbf{1}_{[0,\tau_N]}(s) dB_s \\ &= x + \int_0^t f^{N+1}(X_s^N) \mathbf{1}_{[0,\tau_N]}(s) ds \\ &+ \int_0^t h^{N+1}(X_s^N) \mathbf{1}_{[0,\tau_N]}(s) d\langle B, B \rangle_s + \int_0^t g^{N+1}(X_s^N) \mathbf{1}_{[0,\tau_N]}(s) dB_s \\ &= x + \int_0^{t \wedge \tau_N} f^{N+1}(X_s^N) ds \\ &+ \int_0^{t \wedge \tau_N} h^{N+1}(X_s^N) d\langle B, B \rangle_s + \int_0^{t \wedge \tau_N} g^{N+1}(X_s^N) dB_s, \ 0 \leq t \leq T, \ q.s.. \end{aligned}$$

On the other hand, by the definition of X^{N+1}, we have

$$\begin{aligned} X_{t \wedge \tau_N}^{N+1} = x + \int_0^{t \wedge \tau_N} f^{N+1}(X_s^{N+1}) ds + \int_0^{t \wedge \tau_N} h^{N+1}(X_s^{N+1}) d\langle B, B \rangle_s \\ + \int_0^{t \wedge \tau_N} g^{N+1}(X_s^{N+1}) dB_s, \ 0 \leq t \leq T, \ q.s.. \end{aligned}$$

By the uniqueness of the solution to the GSDE with coefficients f^{N+1}, h^{N+1} and g^{N+1}, X^N and X^{N+1} are distinguishable (in the q.s. sense) on $[0,\tau_N]$. This implies that the sequence $\{\tau_N\}_{N\in\mathbb{N}}$ are q.s. increasing.

Now we aim to show that

$$\bar{C}\bigg(\bigcup_{N=1}^{+\infty}\{\omega:\tau_N(\omega)=T\}\bigg)=1. \tag{2.29}$$

Because $|X^N_\cdot|$ never exceeds N on $[0,\tau_N]$, q.s., we have

$$\begin{aligned} f(X_t^N)\mathbf{1}_{[0,\tau_N]}(t)=f^N(X_t^N)\mathbf{1}_{[0,\tau_N]}(t);\, h(X_t^N)\mathbf{1}_{[0,\tau_N]}(t)=h^N(X_t^N)\mathbf{1}_{[0,\tau_N]}(t); \\ g(X_t^N)\mathbf{1}_{[0,\tau_N]}(t)=g^N(X_t^N)\mathbf{1}_{[0,\tau_N]}(t),\ 0\le t\le T,\ q.s., \end{aligned} \tag{2.30}$$

where the right-hand side are $M_*^p([0,T];\mathbb{R}^n)$ processes. One can see that

$$\begin{aligned} X_{t\wedge\tau_N}^N &= x+\int_0^{t\wedge\tau_N} f^N(X_s^N)ds+\int_0^{t\wedge\tau_N} h^N(X_s^N)d\langle B,B\rangle_s+\int_0^{t\wedge\tau_N} g^N(X_s^N)dB_s \\ &= x+\int_0^{t\wedge\tau_N} f^N(X_s^N)\mathbf{1}_{[0,\tau_N]}(s)ds+\int_0^{t\wedge\tau_N} h^N(X_s^N)\mathbf{1}_{[0,\tau_N]}(s)d\langle B,B\rangle_s \\ &+\int_0^{t\wedge\tau_N} g^N(X_s^N)\mathbf{1}_{[0,\tau_N]}(s)dB_s \\ &= x+\int_0^{t} f(X_s^N)\mathbf{1}_{[0,\tau_N]}(s)ds+\int_0^{t} h(X_s^N)\mathbf{1}_{[0,\tau_N]}(s)d\langle B,B\rangle_s \\ &+\int_0^{t} g(X_s^N)\mathbf{1}_{[0,\tau_N]}(s)dB_s,\ 0\le t\le T,\ q.s.. \end{aligned}$$

Then, apply G-Itô's formula (cf. Theorem 5.4 in Li and Peng [52]) to $\Phi(t\wedge\tau_N,X_{t\wedge\tau_N}^N):=\exp(-C_{LY}(t\wedge\tau_N))V(t\wedge\tau_N,X_{t\wedge\tau_N}^N)$. Since $\mathcal{L}V\le C_{LY}V$ and $X_\cdot^N\equiv X_{\cdot\wedge\tau_N}^N$ on $[0,\tau_N]$, we have for each $t\in[0,T]$, $\mathcal{L}\Phi(t\wedge\tau_N,X_{t\wedge\tau_N}^N)\le 0$. From (2.30) and the fact that $\partial_x V(t,x)$ is Lipschitz continuous and bounded on $B(0,N)$, it is readily observed that $\nabla\Phi(\cdot,X_\cdot^N)g(X_\cdot^N)\mathbf{1}_{[0,\tau_N]}(\cdot)\in M_*^p([0,T])$ (cf. Proposition 3.11 in Li and Peng [52]). Then, we obtain

$$\bar{\mathbb{E}}\bigg[\int_0^{T\wedge\tau_N}\nabla\Phi(t,X_t^N)g(X_t^N)dB_t\bigg]=0.$$

Hence,

$$\bar{\mathbb{E}}[\Phi(T\wedge\tau_N,X_{T\wedge\tau_N}^N)]-\Phi(0,x)=\bar{\mathbb{E}}\bigg[\int_0^T\mathcal{L}\Phi(t\wedge\tau_N,X_{t\wedge\tau_N}^N)dt\bigg]\le 0,$$

and then,

$$\bar{\mathbb{E}}[V(T\wedge\tau_N,X_{T\wedge\tau_N}^N)]\le V(0,x)\exp(C_{LY}T). \tag{2.31}$$

In particular, we have

$$\bar{\mathbb{E}}[\mathbf{1}_{\{\tau_N<T\}}V(T\wedge\tau_N,X_{T\wedge\tau_N}^N)]\le V(0,x)\exp(C_{LY}T).$$

Since X^N has t-continuous paths outside a polar set, $\tau_N<T$ implies $|X_{T\wedge\tau_N}^N|=N$, q.s., from which we deduce

$$\bar{C}(\{\omega:\tau_N(\omega)<T\})\inf_{|x|\ge N}\inf_{t\in[0,T]}V(t,x)\le V(0,x)\exp(C_{LY}T).$$

As $N\to+\infty$, by (H3"), we obtain

$$1\ge\lim_{N\to+\infty}\bar{C}(\{\omega:\tau_N(\omega)=T\})\ge 1-\lim_{N\to+\infty}\bar{C}(\{\omega:\tau_N(\omega)<T\})=1.$$

Since $\{\omega:\tau_N(\omega)=T\}$ is increasing, the upwards convergence theorem (Theorem 2.4) yields (2.29).

Therefore, there exists a polar set A, such that for all $\omega\in A^c$, the following assertion holds: one can find an $N_0(\omega)$ that depends on ω, such that for all $N\ge N_0(\omega)$, $N\in\mathbb{N}$, $\tau_N(\omega)=T$. Then, we define

$$X_t(\omega)=\begin{cases} X_t^{N_0(\omega)}(\omega),\ 0\le t\le T\ ,\ \omega\in A^c; \\ 0\ ,\ \omega\in A. \end{cases} \tag{2.32}$$

From the argument above, we have for each τ_N, $X\mathbf{1}_{[0,\tau_N]} = X^N\mathbf{1}_{[0,\tau_N]} \in M_*^p([0,T];\mathbb{R}^n)$ and thus, $X \in M_w^p([0,T];\mathbb{R}^n)$. Moreover,

$$\begin{aligned}
X_{t\wedge\tau_N} = X^N_{t\wedge\tau_N} &= x + \int_0^{t\wedge\tau_N} f^N(X_s^N)ds + \int_0^{t\wedge\tau_N} h^N(X_s^N)d\langle B,B\rangle_s + \int_0^{t\wedge\tau_N} g^N(X_s^N)dB_s \\
&= x + \int_0^{t\wedge\tau_N} f(X_s)ds + \int_0^{t\wedge\tau_N} h(X_s)d\langle B,B\rangle_s \\
&+ \int_0^{t\wedge\tau_N} g(X_s)dB_s,\ 0\le t\le T,\ q.s.,
\end{aligned}$$

which implies that X satisfies (2.23).

Similarly to (2.31), we have for a fixed $t \in [0,T]$,

$$\bar{\mathbb{E}}[V(t\wedge\tau_N, X^N_{t\wedge\tau_N})] \le V(0,x)\exp(C_{LY}T).$$

Letting $N \to +\infty$, (2.29) and (2.32) yield that $V(t\wedge\tau_N, X^N_{t\wedge\tau_N}) \to V(t,X_t)$, q.s., then Fatou's Lemma (Lemma 2.5) gives

$$\bar{\mathbb{E}}[V(t,X_t)] \le V(0,x)\exp(C_{LY}T).$$

For the issue of uniqueness, we can follow exactly the procedure stated in the proof of Theorem 2.37. The proof is complete. □

Remark 2.39 *During the proof, we have used an extended notion of G-Itô's formula in the following form:*

$$\begin{aligned}
\Phi(t\wedge\tau, X_{t\wedge\tau}) - \Phi(0,x_0) &= \int_0^{t\wedge\tau} (\partial_t\Phi(s,X_s) + \partial_{x^\nu}\Phi(s,X_s)f_s^\nu)ds + \int_0^{t\wedge\tau} \partial_{x^\nu}\Phi(s,X_s)g_s^{\nu j}dB_s^j \\
&+ \int_0^{t\wedge\tau}\left(\partial_{x^\nu}\Phi(s,X_s)h_s^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(s,X_s)g_s^{\mu i}g_s^{\nu j}\right)d\langle B^i,B^j\rangle_s,\ q.s.,
\end{aligned} \tag{2.33}$$

where τ is a stopping time. In fact, if τ equals to the deterministic time T, the two sides of (2.33) equal q.s. as a result of Theorem 5.4 in Li and Peng [52]. Furthermore, both sides of (2.33) have t-continuous paths outside a polar set, then they are distinguishable (in the q.s. sense). Therefore, for any bounded stopping time $t\wedge\tau$, (2.33) holds with no problem.

Remark 2.40 *In Li [50], scalar valued locally Lipschitz GSDEs are also discussed, but under the following linear growth condition: for some $K > 0$,*

$$xb(t,x) + xh(t,x) + |g(t,x)|^2 \le K(1+x^2).$$

One can verify that in this special case, our assumption (H2") is satisfied with $V(\cdot,x) = C_K(1+x^2)$, where C depends on K. In fact, (H2") allows us to consider some GSDEs with polynomial growth coefficients. Here is an example (Duffing and Van der Pol oscillators in random mechanics, cf. Arnold [1] for more examples): letting $n=2$ and $d=1$, we set $h\equiv 0$ and

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} Y_t \\ -\alpha X_t - \beta X_t^3 - \gamma Y_t \end{pmatrix}dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix}dB_t,$$

where α, β, γ and σ are positive constants. In this case, the Lyapunov function could be

$$V(t,x,y) = 1 + \frac{1}{2}y^2 + \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4,\ 0\le t\le T.$$

Remark 2.41 *This result can be generalized to the case where the coefficients are not time homogeneous and are also random, but (H1') should be assumed in additional to (H2") and (H3"). To prove this case, τ_N should be replaced by $\tau_N \wedge \sigma_N$, and then everything can go in a similar way.*

2.6 Appendix

In this appendix, one dimensional G-Itô's formula is proved when a bounded variation process appears in the dynamic and the condition boundedness on the derivatives of Φ is always required. The corresponding results for a G-Itô process X can be found in Li and Peng [52] as Lemma 5.1 - 5.3. For the simplicity of notation, we always assume $C > 0$ is a constant whose value can vary from line to line.

Lemma 2.42 *Let $0 \leq s \leq t \leq T$, f, h and g be elements in $B_b(\Omega_s)$ and K be a bounded element in $M_{FV}([0,T])$ satisfying that for a positive constant α and each $0 \leq u_1 \leq T$,*

$$\lim_{u_2 \to u_1} \bar{\mathbb{E}}[|K_{u_2} - K_{u_1}|^\alpha] = 0. \tag{2.34}$$

Let $\Phi \in \mathcal{C}^2(\mathbb{R})$ be a real function with bounded and Lipschitz first and second order derivatives. We joint a simple G-Itô process and such a K together:

$$X_t - X_s := f \cdot (t-s) + h \cdot (\langle B \rangle_t - \langle B \rangle_s) + g \cdot (B_t - B_s) + K_t - K_s, \tag{2.35}$$

then we have

$$\begin{aligned}\Phi(X_t) - \Phi(X_s) = &\int_s^t \frac{d\Phi}{dx}(X_u) f du + \int_s^t \left(\frac{d\Phi}{dx}(X_u) h + \frac{1}{2}\frac{d^2\Phi}{dx^2}(X_u) g^2 \right) d\langle B \rangle_u \\ &+ \int_s^t \frac{d\Phi}{dx}(X_u) g dB_u + \int_s^t \frac{d\Phi}{dx}(X_u) dK_u, \ q.s..\end{aligned}$$

Remark 2.43 *We would like to explain a little more about our assumptions on K, especially the condition (2.34) that is stronger than pathwise continuity. In this theorem, we assume that $K \in M_{FV}([0,T])$, then by the continuity of paths, we know that K is progressively measurable and thus, for each $t \in [0,T]$, $K_t \in L^0(\Omega_t)$. From the boundedness of K, we can conclude that K_t is an element in $B_b(\Omega_t)$. For each $N \in \mathbb{N}$, we set $\delta = (t-s)/2^N$ and take the partition*

$$\pi_{[s,t]}^{2^N} = \{t_0^{2^N}, t_1^{2^N} \dots, t_{2^N}^{2^N}\} = \{s, s+\delta, \dots, s + 2^N\delta = t\}$$

and

$$K_u^{2^N} := \sum_{k=0}^{2^N-1} K_{t_k} \mathbf{1}_{[t_k, t_{k+1})}(u), \ s \leq u \leq t,$$

which is a well defined process in $M_b^0([0,T])$. To prove this lemma through approximation by simple processes, a key point is that, we should be able to approximate K by K^{2^N} under the norm (2.4), for some $p \geq 2$. As we have already stated in Lin [54], this can be achieved in the classical framework since the dominated convergence theorem is available, however, in the G-framework, we lack such a theorem and we need this stronger condition.

Denoting by $\hat{M}$ the bound of K, for each $N \in \mathbb{N}$, we calculate

$$\begin{aligned}||K^{2^N} - K||_{M_G^p([0,T])} &= \bar{\mathbb{E}}\left[\int_s^t \left(\sum_{k=0}^{2^N-1} |K_u - K_{t_k^{2^N}}|^p \mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(u) \right) du\right] \\ &\leq \int_s^t \left(\sum_{k=0}^{2^N-1} \bar{\mathbb{E}}[|K_u - K_{t_k^{2^N}}|^p] \mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(u) \right) du.\end{aligned}$$

Then, if $p \geq \alpha$, we have

$$||K^{2^N} - K||_{M_G^p([0,T])} \leq (2\hat{M})^{p-\alpha} \int_s^t \left(\sum_{k=0}^{2^N-1} \bar{\mathbb{E}}[|K_u - K_{t_k^{2^N}}|^\alpha] \mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(u) \right) du \to 0, \text{ as } N \to +\infty;$$

otherwise,

$$||K^{2^N} - K||_{M_G^p([0,T])} \leq \int_s^t \left(\sum_{k=0}^{2^N-1} \bar{\mathbb{E}}[|K_u - K_{t_k^{2^N}}|^\alpha]^{p/\alpha} \mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(u) \right) du \to 0, \text{ as } N \to +\infty,$$

where the last convergence is deduced by Lebesgue's dominated convergence theorem. One can easily see that, as a consequence of the convergence above, $K \in M_*^p([0,T])$.

Proof of Lemma 2.42 : Using notation in Lin [54], we denote by M^X the G-Itô part of (2.35), i.e.,

$$M_t^X - M_s^X := f \cdot (t-s) + h \cdot (\langle B \rangle_t - \langle B \rangle_s) + g \cdot (B_t - B_s).$$

We consider always the partition $\pi_{[s,t]}^{2^N}$, then from the second order Taylor expansion, we have

$$\begin{aligned}
\Phi(X_t) - \Phi(X_s) &= \sum_{k=0}^{2^N-1} (\Phi(X_{t_{k+1}^{2^N}}) - \Phi(X_{t_{k+1}^{2^N}})) \\
&= \sum_{k=0}^{2^N-1} \frac{d\Phi}{dx}(X_{t_k^{2^N}})(M_{t_{k+1}^{2^N}}^X - M_{t_k^{2^N}}^X) + \frac{1}{2}\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})(M_{t_{k+1}^{2^N}}^X - M_{t_k^{2^N}}^X)^2 \\
&+ \sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(\xi_k^{2^N})(M_{t_{k+1}^{2^N}}^X - M_{t_k^{2^N}}^X)(K_{t_{k+1}^{2^N}} - K_{t_k^{2^N}}) \\
&+ \frac{1}{2}\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(\xi_k^{2^N})(K_{t_{k+1}^{2^N}} - K_{t_k^{2^N}})^2 \qquad (2.36) \\
&+ \frac{1}{2}\sum_{k=0}^{2^N-1} \left(\frac{d^2\Phi}{dx^2}(\xi_k^{2^N}) - \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\right)(M_{t_{k+1}^{2^N}}^X - M_{t_k^{2^N}}^X)^2 \\
&+ \sum_{k=0}^{2^N-1} \frac{d\Phi}{dx}(X_{t_k^{2^N}})(K_{t_{k+1}^{2^N}} - K_{t_k^{2^N}}) \\
&= I_1^N + I_2^N + I_3^N + I_4^N + I_5^N + I_6^N,
\end{aligned}$$

where $\xi_k^{2^N}$ satisfies $X_{t_k^{2^N}} \wedge X_{t_{k+1}^{2^N}} \le \xi_k^{2^N} \le X_{t_k^{2^N}} \vee X_{t_{k+1}^{2^N}}$.

We have already proved in Remark 2.43, $K^{2^N} \to K$ in $M_*^p([0,T])$, for any $p \geq 2$. Now, we consider $(M^X)^{2^N}$ and because of the boundedness of f, h and g, we can see that $(M^X)^{2^N} \to M^X$ from the following convergence, which can be verified by the BDG type inequalities, where $\tilde{M}$ is the bound of f, h and g:

$$\begin{aligned}
\bar{\mathbb{E}}\Big[\int_s^t |M_u^{2^N} - M_u|^p du\Big] &= \bar{\mathbb{E}}\Big[\int_s^t \Big|\sum_{k=0}^{2^N-1} M_{t_k^{2^N}}^X \mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(u) - M_u^X\Big|^p du\Big] \\
&\le C\tilde{M}^p \int_s^t \left(\sum_{k=0}^{2^N-1} ((u-t_k)^{p/2} + (u-t_k)^p)\mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(u)\right) du \\
&\le C\tilde{M}^p \sum_{k=0}^{2^N-1} \left(\frac{2}{p+2}(t_{k+1}^{2^N} - t_k^{2^N})^{p/2+1} + \frac{1}{p+1}(t_{k+1}^{2^N} - t_k^{2^N})^{p+1}\right) \\
&\le C\tilde{M}^p (t-s)(\delta^{p/2} + \delta^p) \to 0, \text{ as } N \to +\infty.
\end{aligned}$$

Therefore, $X^{2^N} \to X$ in $M_*^p([0,T])$ and thus, the Lipschitz assumption on $\frac{d\Phi}{dx}$ and $\frac{d^2\Phi}{dx^2}$ implies that, in $M_*^p([0,T])$,

$$\sum_{k=0}^{2^N-1} \frac{d\Phi}{dx}(X_{t_k^{2^N}})\mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(\cdot) \to \frac{d\Phi}{dx}(X_.)$$

and

$$\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\mathbf{1}_{[t_k^{2^N}, t_{k+1}^{2^N})}(\cdot) \to \frac{d^2\Phi}{dx^2}(X_.).$$

By considering the definitions for G-stochastic integrals and that all the coefficients f, h and g are bounded, we have, in $L_*^p(\Omega_t)$,

$$I_1 \to \int_s^t \frac{d\Phi}{dx}(X_u) f du + \int_s^t \frac{d\Phi}{dx}(X_u) h d\langle B\rangle_u + \int_s^t \frac{d\Phi}{dx}(X_u) g dB_u.$$

Denoting by $\bar{M}$ the bound of $\frac{d^2\Phi}{dx^2}$, as $N \to 0$,

$$\bar{\mathbb{E}}\Big[\Big|\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}}) f^2 \cdot (t_{k+1}^{2^N} - t_k^{2^N})^2\Big|^p\Big] \le \tilde{M}^{2p}\bar{M}^p (t-s)^p \delta^p \to 0,$$

By Hölder's inequality, we have

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\Big|\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}}) fg \cdot (t_{k+1}^{2^N} - t_k^{2^N})(B_{t_{k+1}^{2^N}} - B_{t_k^{2^N}})\Big|^p\Big] \\
&\qquad \le \tilde{M}^{2p}\bar{M}^p \delta^p \bar{\mathbb{E}}\Big[\Big(\sum_{k=0}^{2^N-1} |B_{t_{k+1}^{2^N}} - B_{t_k^{2^N}}|\Big)^p\Big] \\
&\qquad \le \tilde{M}^{2p}\bar{M}^p \delta^p (2^N)^{p-1} \bar{\mathbb{E}}\Big[\sum_{k=0}^{2^N-1} |B_{t_{k+1}^{2^N}} - B_{t_k^{2^N}}|^p\Big] \\
&\qquad \le \tilde{M}^{2p}\bar{M}^p (t-s)^p \overline{\sigma}^p \delta^{p/2} \to 0,
\end{aligned}$$

Recalling (2.13), one can see that

$$\begin{aligned}
&\Big|\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}}) fh \cdot (t_{k+1}^{2^N} - t_k^{2^N})(\langle B\rangle_{t_{k+1}^{2^N}} - \langle B\rangle_{t_k^{2^N}})\Big| \\
&\qquad \le \overline{\sigma}^2 \sum_{k=0}^{2^N-1} \Big|\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\Big| |f||h| (t_{k+1}^{2^N} - t_k^{2^N})^2, \ q.s.; \\
&\Big|\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}}) h^2 \cdot (\langle B\rangle_{t_{k+1}^{2^N}} - \langle B\rangle_{t_k^{2^N}})^2\Big| \\
&\qquad \le \overline{\sigma}^4 \sum_{k=0}^{2^N-1} \Big|\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\Big| |h|^2 (t_{k+1}^{2^N} - t_k^{2^N})^2, \ q.s.; \\
&\Big|\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}}) hg \cdot (\langle B\rangle_{t_{k+1}^{2^N}} - \langle B\rangle_{t_k^{2^N}})(B_{t_{k+1}^{2^N}} - B_{t_k^{2^N}})\Big| \\
&\qquad \le \overline{\sigma}^2 \sum_{k=0}^{2^N-1} \Big|\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\Big| |h||g| |t_{k+1}^{2^N} - t_k^{2^N}| |B_{t_{k+1}^{2^N}} - B_{t_k^{2^N}}|, \ q.s..
\end{aligned}$$

in which the right-hand sides converge to 0 in $L_*^p(\Omega_t)$. On the other hand, we calculate

$$\begin{aligned}
\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}}) g^2 \cdot (B_{t_{k+1}^{2^N}} - B_{t_k^{2^N}})^2 &= \sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}}) g^2 \cdot (\langle B\rangle_{t_{k+1}^{2^N}} - \langle B\rangle_{t_k^{2^N}}) \\
&\quad - 2\sum_{k=0}^{2^N-1} \frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}}) g^2 \cdot \Big(B_{t_k^{2^N}}(B_{t_{k+1}^{2^N}} - B_{t_k^{2^N}}) - \int_{t_k^{2^N}}^{t_{k+1}^{2^N}} B_u dB_u\Big), \ q.s..
\end{aligned} \tag{2.37}$$

By proposition 3.5 in Li and Peng [52],

$$
\begin{aligned}
&\bar{\mathbb{E}}\Big[\Big|\sum_{k=0}^{2^N-1}\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})g^2\cdot\Big(B_{t_k^{2^N}}(B_{t_{k+1}^{2N}}-B_{t_k^{2^N}})-\int_{t_k^{2^N}}^{t_{k+1}^{2^N}}B_u dB_u\Big)\Big|^p\Big]\\
&\quad=\bar{\mathbb{E}}\Big[\Big|\int_s^t\Big(\sum_{k=0}^{2^N-1}\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})g^2\cdot(B_u-B_{t_k^{2^N}})\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(u)\Big)dB_u\Big|^p\Big]\\
&\quad\le C_p\bar{\sigma}^p\bar{\mathbb{E}}\Big[\Big|\int_s^t\Big(\Big|\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\Big|^2|g|^4|B_u-B_{t_k^{2^N}}|^2\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(u)\Big)du\Big|^{p/2}\Big]\\
&\quad\le C_p\bar{\sigma}^p(t-s)^{p/2-1}\bar{\mathbb{E}}\Big[\int_s^t\Big(\sum_{k=0}^{2^N-1}\Big|\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\Big|^p|g|^{2p}|B_u-B_{t_k^{2^N}}|^p\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(u)\Big)du\Big]\\
&\quad\le C_p'\bar{\sigma}^p(t-s)^{p/2-1}\sum_{k=0}^{2^N-1}\bar{\mathbb{E}}\Big[\int_{t_k^{2^N}}^{t_{k+1}^{2^N}}\Big(\Big|\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})\Big|^p|g|^{2p}|B_u-B_{t_k^{2^N}}|^p\Big)du\Big]\\
&\quad\le C_p''\bar{\sigma}^{2p}\tilde{M}^{2p}\bar{M}^p(t-s)^{p/2}\delta^{p/2}\to 0,
\end{aligned}
$$

from which the second term of (2.37) vanishes. Because, in $M_*^p([0,T])$,

$$
\sum_{k=0}^{2^N-1}\frac{d^2\Phi}{dx^2}(X_{t_k^{2^N}})g^2\mathbf{1}_{[t_k^{2^N},t_{k+1}^{2^N})}(\cdot)\to\frac{d^2\Phi}{dx^2}(X_\cdot)g^2,
$$

similarly to (2.14), we have, in $L_*^p(\Omega_t)$, the first term in (2.37) converges to $\int_s^t\frac{d^2\Phi}{dx^2}(X_u)g^2d\langle B\rangle_u$. In conclusion, in $L_*^p(\Omega_t)$,

$$
I_2\to\frac{1}{2}\int_s^t\frac{d^2\Phi}{dx^2}(X_u)g^2d\langle B\rangle_u.
$$

Since $\frac{d^2\Phi}{dx^2}$ is bounded, and $M_\cdot^X(\omega)$ and $K_\cdot(\omega)$ are t-continuous and thus, uniformly continuously on $[0,T]$ outside a polar set and $K\in M_{FV}([0,T])$, we can easily obtain that I_3^N and I_4^N q.s. vanish.

For I_5^N, we calculate

$$
\begin{aligned}
|I_5^N|&\le\frac{C_L}{2}\sum_{k=0}^{2^N-1}|\xi_k^{2^N}-X_{t_k^{2^N}}||M_{t_{k+1}^{2^N}}^X-M_{t_k^{2^N}}^X|^2\\
&\le\frac{C_L}{2}\Big(\sum_{k=0}^{2^N-1}|(\xi^1)_k^{2^N}-M_{t_k^{2^N}}^X||M_{t_{k+1}^{2^N}}^X-M_{t_k^{2^N}}^X|^2+\sum_{k=0}^{2^N-1}|(\xi^2)_k^{2^N}-K_{t_k^{2^N}}||M_{t_{k+1}^{2^N}}^X-M_{t_k^{2^N}}^X|^2\Big),
\end{aligned}
$$

where C_L is the Lipschtiz constant of $\frac{d^2\Phi}{dx^2}$, $(\xi^1)_k^{2^N}$ satisfies $M_{t_k^{2^N}}^X\wedge M_{t_{k+1}^{2^N}}^X\le(\xi^1)_k^{2^N}\le M_{t_k^{2^N}}^X\vee M_{t_{k+1}^{2^N}}^X$ and $(\xi^2)_k^{2^N}$ satisfies $K_{t_k^{2^N}}\wedge K_{t_{k+1}^{2^N}}\le(\xi^2)_k^{2^N}\le K_{t_k^{2^N}}\vee K_{t_{k+1}^{2^N}}$, q.s.. For the first part,

$$
\mathbb{E}\Big[\Big(\sum_{k=0}^{2^N-1}|(\xi^1)_k^{2^N}-M_{t_k^{2^N}}^X||M_{t_{k+1}^{2^N}}^X-M_{t_k^{2^N}}^X|^2\Big)^p\Big]\le C_p\bar{\sigma}^{3p}(t-s)^p(\delta^{1/2}+\delta^{2p})\to 0,\ \text{as}\ N\to+\infty.
$$

Like I_3^N and I_4^N, the second part also q.s. vanishes. Thus, $|I_5^N|\to 0$, q.s..

Since paths of X are q.s. t-continuous on $[0,T]$, by Remark 2.31, $I_6^N\to\int_s^t\frac{d\Phi}{dx}(X_u)dK_u$, q.s.. We complete the proof. □

Lemma 2.44 *Let $0\le s\le t\le T$, $\Phi\in\mathcal{C}^2(\mathbb{R})$ be a real function with bounded and Lipschitz first and second order derivatives, f, $h\in M_*^1([0,T])$, $g\in M_*^2([0,T])$, K is a bounded element in $M_{FV}([0,T])$ that satisfies (2.34). Consider*

$$
X_t=X_s+\int_s^t f_u du+\int_s^t h_u d\langle B,B\rangle_u+\int_s^t g_u dB_u+K_t-K_s, \tag{2.38}
$$

then,

$$\Phi(X_t)-\Phi(X_s)=\int_s^t \frac{d\Phi}{dx}(X_u)f_u du+\int_s^t\left(\frac{d\Phi}{dx}(X_u)h_u+\frac{1}{2}\frac{d^2\Phi}{dx^2}(X_u)g_u^2\right)d\langle B\rangle_u \qquad (2.39)$$
$$+\int_s^t \frac{d\Phi}{dx}(X_u)g_u dB_u+\int_s^t \frac{d\Phi}{dx}(X_u)dK_u,\ q.s..$$

Proof: Consider the sequences of step processes $\{f^N\}_{N\in\mathbb{N}}$, $\{h^N\}_{N\in\mathbb{N}}$ and $\{g^N\}_{N\in\mathbb{N}}$ of the following form:

$$\eta_t^N=\sum_{k=0}^{N-1}\xi_k\mathbf{1}_{[t_k,t_{k+1})}(t), \qquad (2.40)$$

where $\xi_k\in B_b(\Omega_t)$, $k=0,\dots,N-1$. We assume that $f^N\to f$ and $h^N\to h$ in $M_*^1([0,T])$ and that $g^N\to g$ in $M_*^2([0,T])$. Set

$$X_t^N=x+\int_0^t f_s^N ds+\int_0^t h_s^N d\langle B\rangle_s+\int_0^t g_s^N dB_s+K_t.$$

By generalized BDG type inequalities (cf. Lemma 4.5 in Lin [53], here we need a BDG type inequality with $p=1$), we have

$$\mathbb{E}[\sup_{t\in[0,T]}|X_t^N-X_t|]\le C\Bigg(\mathbb{E}\Big[\int_0^T|f_t^N-f_t|dt\Big]+\mathbb{E}\Big[\int_0^T|h_t^N-h_t|dt\Big]$$
$$+\mathbb{E}\Big[\Big(\int_0^T|g_t^N-g_t|^2dt\Big)^{1/2}\Big]\Bigg)\to 0,\ \text{as } N\to+\infty.$$

Because all the derivatives of Φ are bounded and Lipschitz, we know that

$$\lim_{N\to+\infty}\bar{\mathbb{E}}[\sup_{t\in[0,T]}|\Phi(X_t^N)-\Phi(X_t)|]\to 0, \qquad (2.41)$$

and for any $p\ge 1$,

$$\lim_{N\to+\infty}\bar{\mathbb{E}}\Big[\sup_{t\in[0,T]}\Big|\frac{\partial\Phi}{\partial x}(X_t^N)-\frac{\partial\Phi}{\partial x}(X_t)\Big|^p\Big]\to 0 \text{ and } \lim_{N\to+\infty}\bar{\mathbb{E}}\Big[\sup_{t\in[0,T]}\Big|\frac{\partial^2\Phi}{\partial x^2}(X_t^N)-\frac{\partial^2\Phi}{\partial x^2}(X_t)\Big|^p\Big]\to 0.$$

Then, we can apply Lemma 2.42 for each $\Phi(X_t^N)$ and follow the procedure in the proof of Lemma 5.2 in Li and Peng [52] to verify that $\frac{\partial\Phi}{\partial x}(X_\cdot^N)g_\cdot^N\to\frac{\partial\Phi}{\partial x}(X_\cdot)g_\cdot$ in $M_*^2([0,T])$, and furthermore $\frac{\partial\Phi}{\partial x}(X_\cdot^N)f_\cdot^N\to\frac{\partial\Phi}{\partial x}(X_\cdot)f_\cdot$, $\frac{\partial\Phi}{\partial x}(X_\cdot^N)h_\cdot^N\to\frac{\partial\Phi}{\partial x}(X_\cdot)h_\cdot$ and $\frac{\partial^2\Phi}{\partial x^2}(X_\cdot^N)(g^N)_\cdot^2\to\frac{\partial^2\Phi}{\partial x^2}(X_\cdot)g_\cdot^2$ in $M_*^1([0,T])$. On the other hand, the boundedness of $V_0^T(K)$ and (2.41) imply that

$$\Big|\int_0^T\Big(\frac{\partial\Phi}{\partial x}(X_t^N)-\frac{\partial\Phi}{\partial x}(X_t)\Big)dK_t\Big|\le\int_0^t\Big|\frac{\partial\Phi}{\partial x}(X_t^N)-\frac{\partial\Phi}{\partial x}(X_t)\Big|d|K|_t$$
$$\le\sup_{0\le t\le T}\Big|\frac{\partial\Phi}{\partial x}(X_t^N)-\frac{\partial\Phi}{\partial x}(X_t)\Big|V_0^T(K)\to 0,\ q.s.,$$

which ends the proof. □

Consider a more general $\Phi\in\mathcal{C}^2(\mathbb{R})$ that satisfying Φ, $\frac{\partial\Phi}{\partial x}$ and $\frac{\partial^2\Phi}{\partial x^2}$ are bounded and uniformly continuous. For such Φ, one can find a sequence $\{\Phi^N\}_{N\in\mathbb{N}}$, such that Φ^N, $\frac{\partial\Phi^N}{\partial x}$ and $\frac{\partial^2\Phi^N}{\partial x^2}$ converge uniformly on $\mathbb{R}$ to Φ, $\frac{\partial\Phi}{\partial x}$ and $\frac{\partial^2\Phi}{\partial x^2}$ respectively. Then, we can apply Lemma 2.44 for each $\Phi^N(X_t)$ and obtain the following Lemma:

Lemma 2.45 *Let $0\le s\le t\le T$, $\Phi\in\mathcal{C}^2(\mathbb{R})$ be a real function satisfying that Φ, $\frac{\partial\Phi}{\partial x}$ and $\frac{\partial^2\Phi}{\partial x^2}$ are bounded and uniformly continuous. We still consider an X in the form of (2.38) with the same condition on f, h, g and K. Then, (2.39) still holds.*

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Chapter 3

Multidimensional Reflected GSDEs

Abstract: In this chapter, we study a class of reflected stochastic differential equations driven by G-Brownian motion in a convex domain with bounded coefficients.

Key words. G-Brownian motion; G-Itô's formula; G-stochastic differential equations; reflecting boundary conditions; convex domain.

AMS subject classifications. 60H10

3.1 Formulation to multidimensional reflected GSDEs and convexity

As we have already introduced in Section 1.1, multidimensional reflected stochastic differential equations has been studied by many authors in the classical framework. In this chapter, we consider multidimensional reflected stochastic differential equations driven by G-Brownian motion (multidimensional reflected GSDEs), and the main results of this chapter are established on the space $M_*^2([0,1];\mathbb{R}^n)$, which is a process space introduced in Chapter 2. Furthermore, we always adopt the Einstein notation of summation.

Assume that $\mathcal{O}$ is a strict subset of $\mathbb{R}^n$, which is open and convex, B is a d-dimensional G-Brownian motion, $\langle B,B\rangle = (\langle B^i,B^j\rangle)_{i,j=1,\ldots,d}$ is the mutual variation matrix of B and $x\in\bar{\mathcal{O}}$. Let f, h and g be functions such that for a fixed $\omega\in\Omega$, $f(\cdot,\cdot)(\omega)=(f^1(\cdot,\cdot)(\omega),\ldots,f^n(\cdot,\cdot)(\omega))^{\mathbf{Tr}}:[0,1]\times\mathbb{R}^n\to\mathbb{R}^n$, $h(\cdot,\cdot)(\omega)=(h_{ij}^\nu(\cdot,\cdot)(\omega))_{i,j=1,\ldots,d}^{\nu=1,\ldots,n}:[0,1]\times\mathbb{R}^n\to\mathbb{R}^{n\times d^2}$ and $g(\cdot,\cdot)(\omega)=(g_j^\nu(\cdot,\cdot)(\omega))_{j=1,\ldots,d}^{\nu=1,\ldots,n}:[0,1]\times\mathbb{R}^n\to\mathbb{R}^{n\times d}$.

Furthermore, we assume that

(H1) For each $x\in\mathbb{R}$, $f^\nu(\cdot,x)$, $h_{ij}^\nu(\cdot,x)$, $g_j^\nu(\cdot,x)\in M_*^2([0,1])$, $\nu=1,\ldots,n$, $i,j=1,\ldots,d$.

(H2) The coefficients f, h and g are Lipschitz in x, i.e., for each $t\in[0,1]$, $x,x'\in\mathbb{R}^n$,

$$|f(t,x)-f(t,x')|+||h(t,x)-h(t,x')||+||g(t,x)-g(t,x')||\le C_L|x-x'|,\ q.s.,$$

where $||\cdot||$ is the Hilbert-Schmidt norm of a matrix. Here, q.s. means this condition holds for all the ω outside a polar set A independent of t.

Consider the following n-dimensional reflected GSDEs:

$$X_t = x+\int_0^t f(s,X_s)ds+\int_0^t h(s,X_s)d\langle B,B\rangle_s+\int_0^t g(s,X_s)dB_s-K_t,\ 0\le t\le 1,\ q.s.. \tag{3.1}$$

We say that a couple of processes (X,K) solves the reflected GSDEs (3.1) if

(i) X and K are $M_*^2([0,1];\mathbb{R}^n)$ processes whose paths are continuous on $[0,1]$ outside a polar set A;

(ii) For $\omega\in A^c$, $X_\cdot(\omega)$ takes values in $\bar{\mathcal{O}}$, $K_\cdot(\omega)$ is of bounded variation on $[0,1]$ and $K_0(\omega)=0$;

(iii) Z is a process satisfying that for $\omega\in A^c$, $Z_\cdot(\omega)$ takes values in $\bar{\mathcal{O}}$ and is continuous, then for any $t\in[0,1]$,

$$\int_0^t (X_t(\omega)-Z_t(\omega))dK_t(\omega)\ge 0,\ \text{for all } \omega\in A^c, \tag{3.2}$$

where the integral on the right-hand side of (3.2) is in the sense of Lebesgue-Stieltjes (cf. Section 2.4 of this thesis for details of these integrals in the G-framework). If the boundary of $\bar{\mathcal{O}}$ is smooth enough, the solution of (3.1) coincides with a standard normal reflected G-diffusion. By Lemma 2.1 in Gegout-Petit and Pardoux [28], we have for all $\omega\in A^c$,

$$\int_0^1 \mathbf{1}_{X_t(\omega)\in\mathcal{O}}dK_t(\omega)=0,$$

and denoting by $V_0^t(K)$ the variation of K over $[0,t]$, $K_t(\omega)=\int_0^t v_s dV_0^t(K)$, where v is the inner normal to $\mathcal{O}$ at $X_t(\omega)$.

We denote by $2\beta(x)$ the gradient of the square of the distance to $\mathcal{O}$, i.e.,

$$\beta(x)=(x-\pi(x))^{\mathbf{Tr}},$$

where $\pi(x)$ is the orthogonal projection on $\bar{\mathcal{O}}$. For the technique adopted in what follows, we introduce some well-known properties of convex sets, which can be also find in Menaldi [63] and Gegout-Petit and Pardoux [28]:

$$(x'-x)^{\mathbf{Tr}}(x-\pi(x))\le 0,\ \text{for all } x\in\mathbb{R}^n \text{ and } x'\in\mathcal{O}; \tag{3.3}$$

$$(x'-x)^{\mathbf{Tr}}(x-\pi(x))\le (x'-\pi(x'))^{\mathbf{Tr}}(x-\pi(x)),\ \text{for all } x,\ x'\in\mathbb{R}^n; \tag{3.4}$$

and there exists a point $a\in\mathcal{O}$ and a positive constant γ_a, such that

$$(x-a)^{\mathbf{Tr}}(x-\pi(x))\ge\gamma_a|x-\pi(x)|,\ \text{for all } x\in\mathbb{R}^n. \tag{3.5}$$

From now on, the point a along with γ_a, which ensure that (3.5) hold, are fixed in the following text.

3.2 Convergence results

We assume in addition to (H1) and (H2) the following condition:

(H3) For all $(t,x) \in [0,1] \times \mathbb{R}^n$, $f^\nu(\cdot,x)$, $h^\nu_{ij}(\cdot,x)$ and $g^\nu_j(\cdot,x)$, $\nu = 1,\dots,n$, $i,j = 1,\dots,d$, are bounded and this uniform bound is denoted by $\widetilde{M}$.

For simplicity of notation, we consider (3.1) when $h \equiv 0$, and a similar result hold for the general case.

Following the penalization method adopted by Menaldi [63], we construct a sequence of G-diffusions:

$$X^\varepsilon_t = x + \int_0^t f(s, X^\varepsilon_s)ds + \int_0^t g(s, X^\varepsilon_s)dB_s - \frac{1}{\varepsilon}\int_0^t \beta(X^\varepsilon_s)dt,\ 0 \le t \le 1,\ q.s.. \tag{3.6}$$

Our aim of this section is to prove the following convergence results, such that the reflected GSDE (3.1) admits at least a solution in $M^p_*([0,T];\mathbb{R}^n)$: for any $p \ge 1$,

$$\bar{\mathbb{E}}[\sup_{0\le t\le 1} |X^\varepsilon_t - X^{\varepsilon'}_t|^p] \to 0,\ \text{as}\ \varepsilon,\ \varepsilon' \to 0; \tag{3.7}$$

$$\bar{\mathbb{E}}\left[\sup_{0\le t\le 1}\left|\frac{1}{\varepsilon}\int_0^t \beta(X^\varepsilon_s)ds - \frac{1}{\varepsilon'}\int_0^t \beta(X^{\varepsilon'}_s)ds\right|^p\right] \to 0,\ \text{as}\ \varepsilon,\ \varepsilon' \to 0. \tag{3.8}$$

The assumption (H2) and the boundedness of coefficients (H3) implies that for each $x \in \mathbb{R}^n$ and any $p \ge 2$, $f^\nu(\cdot,x)$ and $g^\nu_j(\cdot,x)$, $\nu = 1,\dots,n$, $j = 1,\dots,d$, are in $M^p_*([0,T])$. On the other hand, $\beta(x)$ is a deterministic Lipschitz function, then for each $x \in \mathbb{R}^n$, $\beta(x) \in M^p_*([0,T];\mathbb{R}^n)$. By Theorem 2.35 in Section 2.4 of this thesis, there exists a unique solution of GSDE (3.6) in $M^p_*([0,1];\mathbb{R}^n)$, for any $p \ge 2$, and $f^\nu(\cdot, X^\varepsilon_\cdot)$, $g^\nu_j(\cdot, X^\varepsilon_\cdot) \in M^p_*([0,T])$.

For a positive constant α, consider a function $\Phi(t,x) := e^{-\alpha t}(1+|x-a|^2)^{p/2}$. To obtain the desired convergence results, we proceed in 3 steps. In the sequel, C_p denotes a positive constant that depends only on p, n, d and Γ (the set that generate the G-expectation), and C denotes another positive constant. These two constants may vary from line to line.

Step 1: At this step, we prove some a priori estimate, which is uniform in ε, i.e., for some $p \ge 1$,

$$\bar{\mathbb{E}}[\sup_{0\le t\le 1}|X^\varepsilon_t|^p] + \bar{\mathbb{E}}\left[\left(\frac{1}{\varepsilon}\int_0^1 |\beta(X^\varepsilon_t)|dt\right)^p\right] \le C,\ \text{for all}\ \varepsilon > 0. \tag{3.9}$$

We shall apply G-Itô's formula (cf. Theorem 5.4 in Li and Peng [52]) to $\Phi(t, X^\varepsilon_t)$, before proceeding this, we calculate, for $\mu, \nu = 1,\dots,n$,

$$\partial\Phi_t(t,x) = -\alpha e^{-\alpha t}(1+|x-a|^2)^{p/2},\ \partial\Phi_{x^\mu}(t,x) = pe^{-\alpha t}(x^\mu - a^\mu)(1+|x-a|^2)^{p/2-1}$$

and

$$\partial\Phi_{x^\mu x^\nu}(t,x) = \begin{cases} p(p-2)e^{-\alpha t}(x^\mu - a^\mu)(x^\nu - a^\nu)(1+|x-a|^2)^{p/2-2} & ,\mu \ne \nu; \\ pe^{-\alpha t}(1+|x-a|^2)^{p/2-1} + p(p-2)e^{-\alpha t}(x^\mu - a^\mu)^2(1+|x-a|^2)^{p/2-2} & ,\mu = \nu. \end{cases}$$

For $p \ge 1$, one can easily see that

$$|\partial\Phi_{x^\mu x^\nu}(t,x)| \le p(p+1)e^{-\alpha t}(1+|x-a|^2)^{p/2-1},\ \mu,\ \nu = 1,\dots,n.$$

Fixing an $\varepsilon > 0$, apply G-Itô's formula to $e^{-\alpha t}(1+|X^\varepsilon_t - a|^2)^{p/2}$, then

$$\begin{aligned} e^{-\alpha t}(1+|X^\varepsilon_t - a|^2)^{p/2} &= (1+|x-a|^2)^{p/2} \\ &+ \int_0^t e^{-\alpha s}(1+|X^\varepsilon_s - a|^2)^{p/2}(-\alpha + p(1+|X^\varepsilon_s - a|^2)^{-1}(X^\varepsilon_s - a)^{\mathbf{Tr}} f(s, X^\varepsilon_s))ds \\ &+ \int_0^t pe^{-\alpha s}(1+|X^\varepsilon_s - a|^2)^{p/2-1}(X^\varepsilon_s - a)^{\mathbf{Tr}} g(s, X^\varepsilon_s)dB_s \\ &+ \frac{1}{2}\int_0^t \partial\Phi_{x^\mu x^\nu}(s, X^\varepsilon_s) g^{\mu i}_s(s, X^\varepsilon_s) g^{\nu j}_s(s, X^\varepsilon_s) d\langle B^i, B^j\rangle_s \\ &- \frac{p}{\varepsilon}\int_0^t e^{-\alpha s}(1+|X^\varepsilon_s - a|^2)^{p/2-1}(X^\varepsilon_s - a)^{\mathbf{Tr}}\beta(X^\varepsilon_s)ds \\ &:= (1+|x-a|^2)^{p/2} + I_1 + I_2 + I_3 + I_4. \end{aligned}$$

From (2.13) and (2.15), we have

$$
\begin{aligned}
|I_3| &\leq \frac{1}{2}\int_0^t |\partial\Phi_{x^\mu x^\nu}(s,X_s^\varepsilon)||g_s^{\mu i}(s,X_s^\varepsilon)||g_s^{\nu j}(s,X_s^\varepsilon)|d\langle B^i,B^j\rangle_s \\
&\leq \frac{n^2d^2\tilde{M}^2}{8}\int_0^t p(p+1)e^{-\alpha s}(1+|X_s^\varepsilon-a|^2)^{p/2-1}\max_{i,j=1,\dots,d}(\sigma_{(\mathbf{e}^i+\mathbf{e}^j)(\mathbf{e}^i+\mathbf{e}^j)^{\mathbf{Tr}}}+\sigma_{(\mathbf{e}^i-\mathbf{e}^j)(\mathbf{e}^i-\mathbf{e}^j)^{\mathbf{Tr}}})ds \\
&:= \frac{C_G n^2d^2\tilde{M}^2}{2}\int_0^t p(p+1)e^{-\alpha s}(1+|X_s^\varepsilon-a|^2)^{p/2-1}ds,
\end{aligned}
$$

where $\mathbf{e}^i$ is the ith component of the orthogonal basis of $\mathbb{R}^d$ and $C_G := \frac{1}{4}\max_{i,j=1,\dots,d}(\sigma_{(\mathbf{e}^i+\mathbf{e}^j)(\mathbf{e}^i+\mathbf{e}^j)^{\mathbf{Tr}}}$ $+\sigma_{(\mathbf{e}^i-\mathbf{e}^j)(\mathbf{e}^i-\mathbf{e}^j)^{\mathbf{Tr}}})$. To keep I_1+I_3 always non-positive, it suffices that

$$
\begin{aligned}
\alpha \geq \alpha_p &:= p\left(\frac{\tilde{M}+C_G n^2d^2\tilde{M}^2(p+1)}{2}\right) \\
&\geq \sup_{\substack{t\in[0,1]\\ x\in\mathbb{R}^n}} p(1+|x-a|^2)^{-1}\left((x-a)^{\mathbf{Tr}}f(t,x)+\frac{C_G n^2d^2\tilde{M}^2(p+1)}{2}\right),
\end{aligned}
$$

which is independent of ε. Thus, for $\alpha\geq\alpha_p$,

$$
\begin{aligned}
e^{-\alpha t}(1+|X_t^\varepsilon-a|^2)^{p/2} \leq{}& (1+|x-a|^2)^{p/2} \\
&+\int_0^t pe^{-\alpha s}(1+|X_s^\varepsilon-a|^2)^{p/2-1}(X_s^\varepsilon-a)^{\mathbf{Tr}}g(s,X_s^\varepsilon)dB_s \\
&-\frac{p}{\varepsilon}\int_0^t e^{-\alpha s}(1+|X_s^\varepsilon-a|^2)^{p/2-1}(X_s^\varepsilon-a)^{\mathbf{Tr}}\beta(X_s^\varepsilon)ds.
\end{aligned}
\tag{3.10}
$$

We see that $(e^{-\alpha t})_{0\leq t\leq T}$ and $(g_j(t,X_t^\varepsilon))_{0\leq t\leq T}$ are bounded process in $M_*^2([0,T])$. Similarly to (3.18) in Lin and Bai [56], because X^ε has the moment of any order, $1+|X_\cdot^\varepsilon-a|^2\in M_*^p([0,T])$, for any $p\geq 2$. Define $p' := p/2-1$ and assume $0\leq p'<1$. For a_1, $a_2\geq 1$, the following inequality holds true: $|a_1^{p'}-a_2^{p'}|\leq|a_1-a_2|$. If $\{Y^N\}_{N\in\mathbb{N}}$ is a sequence of step processes in $M_b^0([0,T])$ that approximates $1+|X_\cdot^\varepsilon-a|^2$ in $M_*^2([0,T])$, by the inequality above, we can deduce that $\{(Y^N)^{p'}\}_{N\in\mathbb{N}}$ is also a sequence that approximates $(1+|X_\cdot^\varepsilon-a|^2)^{p/2-1}$ under the same norm. On the other hand, if $p'>1$, using the method in Lin and Bai [56], it is easy to show that $(1+|X_\cdot^\varepsilon-a|^2)^{p/2-1}\in M_*^2([0,T])$. By Proposition 3.11 in Li and Peng [52], we conclude that $(pe^{-\alpha t}(1+|X_t^\varepsilon-a|^2)^{p/2-1}(X_t^\varepsilon-a)^{\mathbf{Tr}}g(t,X_t^\varepsilon))_{0\leq t\leq T}\in M_*^2([0,T])$. Then, taking the G-expectation on both sides, the second term in (3.10) vanishes. From (3.3), we obtain for $p\geq 2$,

$$
\bar{\mathbb{E}}[|X_t^\varepsilon|^p]\leq C,\ 0\leq t\leq 1. \tag{3.11}
$$

Letting $p=2$, from (3.5), we have

$$
\frac{2\gamma_a}{\varepsilon}\int_0^t e^{-\alpha s}|\beta(X_s^\varepsilon)|ds\leq(1+|x-a|^2)+2\int_0^t e^{-\alpha s}(X_s^\varepsilon-a)^{\mathbf{Tr}}g(s,X_s^\varepsilon)dB_s. \tag{3.12}
$$

By the BDG type inequality (2.6), we deduce from (3.12), for $p\geq 2$,

$$
\bar{\mathbb{E}}\left[\left|\frac{1}{\varepsilon}\int_0^1|\beta(X_t^\varepsilon)|dt\right|^p\right]\leq C_p e^{\alpha p}\left(\frac{1}{(2\gamma_a)^p}(1+|x-a|^2)^p+\frac{\tilde{M}^p}{\gamma_a^p}\bar{\mathbb{E}}\left[\left(\int_0^1|X_t^\varepsilon-a|^2dt\right)^{p/2}\right]\right).
$$

Then, from (3.11), we have

$$
\bar{\mathbb{E}}\left[\left|\frac{1}{\varepsilon}\int_0^1|\beta(X_t^\varepsilon)|dt\right|^p\right]\leq C_p e^{\alpha p}\left(\frac{1}{(2\gamma_a)^p}(1+|x-a|^2)^p+\frac{\tilde{M}^p}{\gamma_a^p}\bar{\mathbb{E}}\left[\int_0^1|(X_t^\varepsilon-a)|^pdt\right]\right)\leq C; \tag{3.13}
$$

If $1\leq p<2$, Hölder's inequality yields similar results as (3.11) and (3.13).

Re-considering (3.10) when $p = 2$, with the help of (3.13), we calculate for $q \geq 1$,

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{0\leq t\leq 1}|X_t^\varepsilon|^q] &\leq C_q(\bar{\mathbb{E}}[\sup_{0\leq t\leq 1}(1+|X_t^\varepsilon-a|^2)^{q/2}]+|a|^q)\\ &\leq C_q e^\alpha\bigg((1+|x-a|^2)^{q/2}+|a|^q+\bar{\mathbb{E}}\bigg[\sup_{0\leq t\leq 1}\bigg|\int_0^t e^{-\alpha s}(X_s^\varepsilon-a)^{\mathbf{Tr}}g(s,X_s^\varepsilon)dB_s\bigg|^q\bigg]\bigg)\\ &\leq C_q e^\alpha\bigg((1+|x-a|^2)^{q/2}+|a|^q+\tilde{M}^q\bar{\mathbb{E}}\bigg[\bigg(\int_0^1|X_t^\varepsilon-a|^2dt\bigg)^{q/2}\bigg]\bigg).\end{aligned}$$

By a similar argument for (3.13), it follows that for $p \geq 1$,

$$\bar{\mathbb{E}}[\sup_{0\leq t\leq 1}|X_t^\varepsilon|^p]\leq C,$$

where $C > 0$ is independent of ε and depends only on p, n, d, Γ, x, a and $\tilde{M}$, from which and (3.13), (3.9) is obtained.

Step 2: At this step, we prove that for any $p > 2$, there exists a $C > 0$ that is independent of ε, such that

$$\bar{\mathbb{E}}[\sup_{0\leq t\leq 1}|\beta(X_t^\varepsilon)|^p]\leq C\varepsilon^{p/2-1},\ \text{for all } \varepsilon>0. \tag{3.14}$$

Consider a function $\phi(x) := |\beta(x)|^p$, $p > 2$, which is twice continuously differentiable. We calculate

$$|\nabla\phi(x)\cdot f(t,x)| = p|\beta(x)|^{p-2}|\beta(x)\cdot f(t,x)|\leq n\tilde{M}p|\beta(x)|^{p-1},$$

and for μ, $\nu = 1, \dots, n$, and a Δh^ν that contributes to the νth component of x,

$$\begin{aligned}&\frac{||\beta(x+\Delta h^\nu)|^{p-2}(x^\mu-(\pi(x+\Delta h^\nu))^\mu)-|\beta(x)|^{p-2}(x^\mu-(\pi(x))^\mu)|}{\Delta h^\nu}\\ &\leq\frac{|\beta(x)|^{p-2}|(\pi(x+\Delta h^\nu))^\mu-(\pi(x))^\mu)|}{\Delta h^\nu}+|x^\mu-(\pi(x))^\mu|\frac{||\beta(x+\Delta h^\nu)|^{p-2}-|\beta(x)|^{p-2}|}{\Delta h^\nu}\\ &\leq\frac{|\beta(x)|^{p-2}|(\pi(x+\Delta h^\nu))-(\pi(x)))|}{\Delta h^\nu}+2|x^\mu-(\pi(x))^\mu||x^\nu-(\pi(x))^\nu||\beta(x)|^{p-4}+o(\Delta h^\nu)\\ &\leq 3|\beta(x)|^{p-2}+o(\Delta h^\nu),\end{aligned}$$

from which we have $|\partial^2_{x^\mu x^\nu}\phi(x)|\leq 3|\beta(x)|^{p-2}$. Applying G-Itô's formula to $\phi(X_t^\varepsilon)$, we obtain

$$\begin{aligned}|\beta(X_t^\varepsilon)|^p+\frac{p}{\varepsilon}\int_0^t|\beta(X_s^\varepsilon)|^pds &\leq\int_0^t\bigg(n\tilde{M}p|\beta(X_s^\varepsilon)|^{p-1}+\frac{3C_Gn^2d^2\tilde{M}^2p}{2}|\beta(X_s^\varepsilon)|^{p-2}\bigg)ds\\ &\quad+\int_0^t p|\beta(X_s^\varepsilon)|^{p-2}\beta(X_s^\varepsilon)^{\mathbf{Tr}}g(s,X_s^\varepsilon)dB_s\\ &\leq C\int_0^t(|\beta(X_s^\varepsilon)|^{p-1}+|\beta(X_s^\varepsilon)|^{p-2})ds\\ &\quad+\int_0^t p|\beta(X_s^\varepsilon)|^{p-2}\beta(X_s^\varepsilon)^{\mathbf{Tr}}g(s,X_s^\varepsilon)dB_s.\end{aligned} \tag{3.15}$$

By Young's inequality, we have

$$C|\beta(X_s^\varepsilon)|^{p-1}\leq\frac{p-1}{4\varepsilon}|\beta(X_s^\varepsilon)|^p+\frac{C^p(4\varepsilon)^{p-1}}{p^p}\ \text{and}\ C|\beta(X_s^\varepsilon)|^{p-2}\leq\frac{p-2}{4\varepsilon}|\beta(X_s^\varepsilon)|^p+\frac{2C^{p/2}(4\varepsilon)^{p/2-1}}{p^{p/2}}. \tag{3.16}$$

Putting (3.16) into (3.15), one can see that

$$|\beta(X_t^\varepsilon)|^p+\frac{p}{2\varepsilon}\int_0^t|\beta(X_s^\varepsilon)|^pds\leq Ct(\varepsilon^{p-1}+\varepsilon^{p/2-1})+\int_0^t p|\beta(X_s^\varepsilon)|^{p-2}\beta(X_s^\varepsilon)^{\mathbf{Tr}}g(s,X_s^\varepsilon)dB_s. \tag{3.17}$$

Taking the G-expectation on both side of (3.17), for a similar reason we have showed in Steps 1, we can have

$$\bar{\mathbb{E}}\bigg[\int_0^t|\beta(X_s^\varepsilon)|^pds\bigg]\leq C\varepsilon^{p/2},\ \text{for all } 0<\varepsilon<1.$$

Taking supremum over $[0,1]$ and then taking the G-expectation on both sides of (3.17), by the BDG type inequality (2.6), we have

$$\begin{aligned}
\bar{\mathbb{E}}[\sup_{0\le t\le 1}|\beta(X_t^\varepsilon)|^p] &\le C\varepsilon^{p/2-1} + pd\tilde{M}\bar{\mathbb{E}}\bigg[\bigg(\int_0^1 |\beta(X_t^\varepsilon)|^{2p-2}dt\bigg)^{1/2}\bigg] \\
&\le C\varepsilon^{p/2-1} + pd\tilde{M}\bar{\mathbb{E}}\bigg[\bigg((\sup_{0\le t\le 1}|\beta(X_t^\varepsilon)|^p)\int_0^1 |\beta(X_t^\varepsilon)|^{p-2}dt\bigg)^{1/2}\bigg] \\
&\le C\varepsilon^{p/2-1} + \frac{1}{2}\bar{\mathbb{E}}[\sup_{0\le t\le 1}|\beta(X_t^\varepsilon)|^p] + \frac{p^2d^2\tilde{M}^2}{2}\bar{\mathbb{E}}\bigg[\int_0^1 |\beta(X_t^\varepsilon)|^{p-2}dt\bigg] \qquad (3.18) \\
&\le C\varepsilon^{p/2-1} + \frac{1}{2}\bar{\mathbb{E}}[\sup_{0\le t\le 1}|\beta(X_t^\varepsilon)|^p] + \frac{p^2d^2\tilde{M}^2}{2}\bar{\mathbb{E}}\bigg[\int_0^1 |\beta(X_t^\varepsilon)|^{p}dt\bigg]^{(p-2)/p} \\
&\le C(\varepsilon^{p/2-1} + \bar{\mathbb{E}}[\sup_{0\le t\le 1}|\beta(X_t^\varepsilon)|^p]),
\end{aligned}$$

where $C>0$ is independent of ε and depends only on p, n, d, Γ, x, a and $\tilde{M}$,which implies (3.14).

Step 3: At this step, we prove that for each ε, $\varepsilon'>0$, there exists a $C>0$ that is independent of ε and ε', such that for $2<4q<p<+\infty$,

$$\bar{\mathbb{E}}[\sup_{0\le t\le 1}|X_t^\varepsilon - X_t^{\varepsilon'}|^p] \le C(\varepsilon+\varepsilon')^q. \qquad (3.19)$$

Giving ε, $\varepsilon'>0$, we first consider a intermediate result: for $p'\ge 1$ and $r'>2$, by Hölder's inequality, we calculate

$$\begin{aligned}
\bar{\mathbb{E}}\bigg[\bigg(\frac{1}{\varepsilon'}\int_0^1 |\beta(X_t^\varepsilon)^{\mathbf{Tr}}\beta(X_t^{\varepsilon'})|dt\bigg)^{p'}\bigg] &\le \bar{\mathbb{E}}\bigg[(\sup_{0\le t\le 1}|\beta(X_t^\varepsilon)|^{p'})\bigg(\frac{1}{\varepsilon'}\int_0^1 |\beta(X_t^{\varepsilon'})|dt\bigg)^{p'}\bigg] \\
&\le \bar{\mathbb{E}}[\sup_{0\le t\le 1}|\beta(X_t^\varepsilon)|^{r'p'}]^{1/r'}\bar{\mathbb{E}}\bigg[\bigg(\frac{1}{\varepsilon'}\int_0^1 |\beta(X_t^{\varepsilon'})|dt\bigg)^{\frac{r'p'}{r'-1}}\bigg]^{\frac{(r'-1)}{r'}} \qquad (3.20) \\
&\le C\varepsilon^{p'/2-1/r'},
\end{aligned}$$

where the last inequality is obtained from (3.9) and (3.14). Letting r be sufficiently large, we conclude that for any $0<2q'<p'$, there exists a $C>0$, such that the following statement holds:

$$\bar{\mathbb{E}}\bigg[\bigg(\frac{1}{\varepsilon'}\int_0^1 |\beta(X_t^\varepsilon)^{\mathbf{Tr}}\beta(X_t^{\varepsilon'})|dt\bigg)^{p'}\bigg] \le C\varepsilon^{q'}. \qquad (3.21)$$

Now we start to prove (3.19). Applying G-Itô's formula to $|X_t^\varepsilon - X_t^{\varepsilon'}|^2$, we obtain

$$\begin{aligned}
|X_t^\varepsilon - X_t^{\varepsilon'}|^2 \le{}& nC_L(2+d^2C_LC_G)\int_0^t |X_s^\varepsilon - X_s^{\varepsilon'}|^2ds + 2\int_0^t (X_s^\varepsilon - X_s^{\varepsilon'})^{\mathbf{Tr}}(g(s,X_s^\varepsilon)-g(s,X_s^{\varepsilon'}))dB_s \\
&+ \frac{1}{\varepsilon}\int_0^t (X_s^\varepsilon - X_s^{\varepsilon'})^{\mathbf{Tr}}\beta(X_s^\varepsilon)ds + \frac{1}{\varepsilon'}\int_0^t (X_s^\varepsilon - X_s^{\varepsilon'})^{\mathbf{Tr}}\beta(X_s^{\varepsilon'})ds.
\end{aligned}$$

Similarly to (3.18), we have the following estimates for the G-Itô type integral on the right-hand side of the inequality above: for $r>1$ and some $C_r>0$ depends on r,

$$\begin{aligned}
&\bar{\mathbb{E}}\bigg[\sup_{0\le s\le t}\bigg(\int_0^s (X_u^\varepsilon - X_u^{\varepsilon'})^{\mathbf{Tr}}(g(u,X_u^\varepsilon)-g(u,X_u^{\varepsilon'}))dB_u\bigg)^r\bigg] \\
&\qquad \le C_rC_L^r\bar{\mathbb{E}}\bigg[\bigg(\int_0^t |X_s^\varepsilon - X_s^{\varepsilon'}|^4ds\bigg)^{r/2}\bigg] \\
&\qquad \le \frac{1}{2}\bar{\mathbb{E}}[\sup_{0\le t\le 1}|X_t^\varepsilon - X_t^{\varepsilon'}|^{2r}] + \frac{C_r^2C_L^{2r}}{2}\bar{\mathbb{E}}\bigg[\bigg(\int_0^t |X_s^\varepsilon - X_s^{\varepsilon'}|^2ds\bigg)^r\bigg].
\end{aligned}$$

Letting $r = p' = p/2$ and $q' = q$, thanks to (3.4) and (3.21), we deduce

$$\begin{aligned}\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s^\varepsilon - X_s^{\varepsilon'}|^p] &\le C\bigg(\int_0^t \bar{\mathbb{E}}[\sup_{0\le u\le s}|X_u^\varepsilon - X_u^{\varepsilon'}|^p]ds \\ &\quad + \bar{\mathbb{E}}\bigg[\Big(\frac{1}{\varepsilon}\int_0^1 |\beta(X_t^\varepsilon)^{\mathbf{Tr}}\beta(X_t^{\varepsilon'})|dt\Big)^{p/2}\bigg] + \bar{\mathbb{E}}\bigg[\Big(\frac{1}{\varepsilon'}\int_0^1 |\beta(X_t^\varepsilon)^{\mathbf{Tr}}\beta(X_t^{\varepsilon'})|dt\Big)^{p/2}\bigg]\bigg) \\ &= C\bigg((\varepsilon+\varepsilon')^q + \int_0^t \bar{\mathbb{E}}[\sup_{0\le u\le s}|X_u^\varepsilon - X_u^{\varepsilon'}|^p]ds\bigg).\end{aligned}$$

Gronwall's inequality implies (3.19). Subsequently, from (3.6),

$$\begin{aligned}\bar{\mathbb{E}}\bigg[\sup_{0\le t\le 1}\bigg|\frac{1}{\varepsilon}\int_0^t \beta(X_s^\varepsilon)ds - \frac{1}{\varepsilon'}\int_0^t \beta(X_s^{\varepsilon'})ds\bigg|^p\bigg] &\le C_p\bigg(\bar{\mathbb{E}}[\sup_{0\le t\le 1}|X_t^\varepsilon - X_t^{\varepsilon'}|^p]) \\ &\quad + C_L^p\int_0^1 \bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s^\varepsilon - X_s^{\varepsilon'}|^p]dt \\ &\quad + \bar{\mathbb{E}}\bigg[\sup_{0\le t\le 1}\bigg|\int_0^t (g(s,X_s^\varepsilon) - g(s,X_s^{\varepsilon'}))dB_s\bigg|^p\bigg]\bigg) \\ &\le C_p(1+C_L^p)(\varepsilon+\varepsilon')^q.\end{aligned}$$

Thus, for $p > 2$, (3.7) and (3.8) are obtained. For $1 \le p < 2$, by Hölder's inequality, (3.7) and (3.8) hold as well. Now, we present our main result in this subsection:

Theorem 3.1 *Assume that (H1)-(H3) hold, then (3.1) admits a unique solution* $(X, K) \in M_*^2([0,1];\mathbb{R}^n) \times (M_{FV}([0,1];\mathbb{R}^n) \times M_*^2([0,1];\mathbb{R}^n))$.

Proof: For $p > 2$, we define that X is the limit of $\{X^\varepsilon\}_{\varepsilon>0}$ in the sense of (3.7) and K is the limit of $\{\frac{1}{\varepsilon}\int_0^\cdot \beta(X_s^\varepsilon)ds\}_{\varepsilon>0}$ in the sense of (3.8). One can verify that

$$\bar{\mathbb{E}}[\sup_{0\le t\le 1}|X_t^\varepsilon - X_t|^p] \le \varepsilon^q \to 0, \text{ and } \bar{\mathbb{E}}\bigg[\sup_{0\le t\le 1}\bigg|\frac{1}{\varepsilon}\int_0^t \beta(X_s^\varepsilon)ds - K_t\bigg|^p\bigg] \le \varepsilon^q \to 0, \text{ as } \varepsilon \to 0,$$

then for any $p \ge 2$, $(X, K) \in M_*^p([0,1];\mathbb{R}^n) \times M_*^p([0,1];\mathbb{R}^n)$. Following a standard argument, we can see that there exists a polar set A, outside which all paths $X_\cdot(\omega)$ and $K_\cdot(\omega)$ are continuous. Also one can find a subsequence $\{X^{\varepsilon_k}\}_{k\in\mathbb{N}}$ such that

$$\sup_{0\le t\le 1}|X_t^{\varepsilon_k} - X_t|^p \to 0, \text{ and } \sup_{0\le t\le 1}\bigg|\frac{1}{\varepsilon_k}\int_0^t \beta(X_s^{\varepsilon_k})ds - K_t\bigg|^p \to 0, \ q.s..$$

From (3.14), there exists a polar set A, outside which $X_t(\omega) \in \bar{\mathcal{O}}$, $0 \le t \le 1$. On the other hand, defining $K_t^\varepsilon = \frac{1}{\varepsilon}\int_0^t \beta(X_s^\varepsilon)ds$, for any partition $\pi_{[0,1]}^N$ on $[0,1]$, by Lemma 2.5 in Section 2.2 of this thesis and from (3.9), we have

$$\begin{aligned}\bar{\mathbb{E}}\bigg[\Big(\sum_{i=0}^{N-1}|K_{t_{i+1}} - K_{t_i}|\Big)^p\bigg] &= \bar{\mathbb{E}}\bigg[\lim_{k\to+\infty}\Big(\sum_{i=0}^{N-1}|K_{t_{i+1}}^{\varepsilon_k} - K_{t_i}^{\varepsilon_k}|\Big)^p\bigg] \\ &\le \liminf_{k\to+\infty}\bar{\mathbb{E}}\bigg[\Big(\sum_{i=0}^{N-1}|K_{t_{i+1}}^{\varepsilon_k} - K_{t_i}^{\varepsilon_k}|\Big)^p\bigg] \\ &\le \liminf_{k\to+\infty}\bar{\mathbb{E}}\bigg[\Big(\frac{1}{\varepsilon_k}\int_0^1 |\beta(X_s^{\varepsilon_k})|dt\Big)^p\bigg] \le C,\end{aligned}$$

which implies that $\bar{\mathbb{E}}[(V_0^T(K))] \le C$ and there exists a polar set A, outside which, $V_0^T(K(\omega)) < +\infty$. Thanks to (3.3), giving any Z satisfying that for all $\omega \in A^c$, $Z(\omega)$ takes values in $\bar{\mathcal{O}}$ and is continuous, for any $t \in [0,1]$,

$$\int_0^t (X_t^{\varepsilon_k} - Z_t)dK_t^{\varepsilon_k} \ge 0, \ 0 \le t \le 1.$$

By Lemma 5.7 in Gegout-Petit and Pardoux [28], we pass to the limit as $k \to +\infty$, then (3.2) is obtained.

We turn to prove the uniqueness and suppose there are two pairs (X, K) and (X', K') that are solutions to (3.1) in $M_*^2([0,1];\mathbb{R}^n) \times M_*^2([0,1];\mathbb{R}^n)$. We apply Itô's formula under each $\mathbb{P} \in \mathcal{P}_G$ to $|X_t - X_t'|^2$, then we have

$$E^{\mathbb{P}}[|X_t - X_t'|^2] \le C\bigg(E^{\mathbb{P}}\bigg[\int_0^t |X_s - X_s'|^2 ds\bigg] + E^{\mathbb{P}}\bigg[\int_0^t (X_s - X_s')d(K_s - K_s')\bigg]\bigg),$$

in which the second term is negative as a result of (3.2). Taking supremum over $\mathcal{P}_G$ on both sides, Gronwall's inequality implies $\bar{\mathbb{E}}[|X_t - X_t'|^2] = 0$, $0 \le t \le 1$. That is to say X and X' are $\bar{C}$-modification of each other, and by the continuity of paths and Proposition 2.12 in this thesis, these two processes are indistinguishable (in the q.s. sense). □

Remark 3.2 *Suppose x and $x' \in \bar{\mathcal{O}}$ are two initial values and for $\alpha \ge nC_L(2 + d^2 C_L C_G) := \alpha_2$, we apply Itô's formula under each $\mathbb{P} \in \mathcal{P}_G$ to $e^{-\alpha t}|X_t - X_t'|^2$, then we obtain similarly to (3.10),*

$$\begin{aligned} e^{-\alpha}|X_1^x - X_1^{x'}|^2 &+ (\alpha - \alpha_2)\int_0^1 e^{-\alpha t}|X_t^x - X_t^{x'}|^2 dt \\ &\le |x - x'|^2 + \int_0^1 p e^{-\alpha t}(X_t^x - X_t^{x'})^{\mathbf{Tr}}(g(t, X_t^x) - g(t, X_t^{x'}))dB_t. \end{aligned}$$

Taking first $E^{\mathbb{P}}[\cdot]$ then the supremum over $\mathcal{P}_G$, we have

$$\bar{\mathbb{E}}\bigg[|X_1^x - X_1^{x'}|^2 + (\alpha - \alpha_2)\int_0^1 |X_t^x - X_t^{x'}|^2 dt\bigg] \le e^{\alpha}|x - x'|^2.$$

Remark 3.3 *We notice that these convergence (3.7) and (3.8) may not uniform in the initial data x, because the constant C depends on the distance of x and a fixed point $a \in \mathcal{O}$. However, if we suppose that $\mathcal{O}$ is bounded, then these convergence results can be uniform in x, which are indispensable for further study to the optimal control of G-diffusions.*

Part II

Second Order Backward Stochastic Differential Equations

Chapter 4

Quadratic Second Order BSDEs

Abstract: In this chapter, we study a class of second order backward stochastic differential equations (2BSDEs) with quadratic growth in coefficients. We first establish the solvability for such 2BSDEs and then give their applications to robust utility maximization problems.

Key words. second order BSDEs; quadratic growth; robust utility maximization.

AMS subject classifications. 60H10; 60H30

A paper that concerns main results of this chapter
was submitted for publication,
under the title:
A New Result for Second Order BSDEs
with Quadratic Growth and its Applications

4.1 Introduction

Typically, nonlinear backward stochastic differential equations (BSDEs) are defined on a Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and of the following type:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s,\ 0 \le t \le T, \tag{4.1}$$

where B is a Brownian motion, $\mathcal{F}$ is the $\mathbb{P}$-augmented natural filtration generated by B, g is a nonlinear generator, T is the terminal time and $\xi \in \mathcal{F}_T$ is the terminal value. A solution to BSDE (4.1) is a couple of processes (Y, Z) adapted to the filtration $\mathcal{F}$ that satisfies (4.1).

Under a Lipschitz condition on the generator g, Pardoux and Peng [68] first provided the wellposedness of (4.1). Since then, the theory of nonlinear BSDEs has been extensively studied in the past twenty years. Among all the contributions, we only quote the results which are highly related to our present work.

A weaker assumption on the generator is that g has a quadratic growth in z. This kind of real valued BSDEs with bounded terminal condition was first examined by Kobylanski [46], who used a wxeak convergence technique borrowed from some PDE literature to prove the existence and also obtained the uniqueness result under some additional condition on g. With the help of contracting mapping principle, Tevzadze [93] re-considered this type of BSDEs when the terminal value ξ is small enough in norm. The advantage of the method adopted by Tevzadze [93] is its applicability to not only one-dimensional quadratic BSDEs but also to multidimensional ones. Particularly, the restriction on ξ can be loosened when g satisfies some restrictive condition on its regularity. Briand and Hu [7, 8] extended the existence result for (4.1) to the case that ξ is not uniformly bounded and provided the uniqueness result when g is convex. Besides, Morlais [64] considered some similar type of BSDEs driven by continuous martingales.

Aiming to provide a new mathematical context for improving the classical expected utility theory based on the linear expectation, Peng [71] defined a so-called g-expectation $\mathcal{E}^g[\xi] := Y_0$ on $L^2(\mathcal{F}_T)$ via nonlinear BSDEs with Lipschitz generator. Also, a conditional expectation can be consistently defined: $\mathcal{E}^g[\xi|\mathcal{F}_t] := Y_t$, under which the solution Y of the BSDE with the generator g is a g-martingale. As the counterparts in the classical framework under a linear expectation, Peng [72] gave the notion of g-supermartingle (g-submartingle) and established the nonlinear Doob-Meyer type decomposition theorem. Subsequently, Chen and Peng [10] proved the downcrossing inequality for g-martingales. For the case that g is allowed to have a quadratic growth in z, similar results can be found in Ma and Yao [59].

Recently, Soner et al. [86] established a framework of "quasi-sure" stochastic analysis under a non-dominated class of probability measures. This provided a new approach for Soner et al. [87, 88] to reconsider the wellposedness of second order BSDEs (2BSDEs) introduced by Cheridito et al. [11]. The key idea in Soner et al. [87] is to reinforce a condition that the following 2BSDE holds true $\mathcal{P}_H$-quasi-surely, i.e., $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$, which is a class of mutually singular probability measures (cf. Definition 4.1):

$$Y_t = \xi + \int_t^1 \hat{F}_s(Y_s, Z_s)ds - \int_t^1 Z_s dB_s + K_1 - K_t,\ 0 \le t \le 1. \tag{4.2}$$

Under a uniformly Lipschitz condition on the generator $\hat{F}$, Soner et al. [87] provided a complete wellposedness result for the 2BSDE (4.2). In this pioneering work, a representation theorem of the solution Y is established and thus, the uniqueness is a straightforward corollary. For the existence, a process Y is pathwisely constructed and verified as a $\hat{F}$-supermartingale under each $\mathbb{P} \in \mathcal{P}_H$. Applying the nonlinear Doob-Meyer decomposition theorem, the right-hand side of (4.2) comes out, where K is a (family of) increasing (but could not be strictly increasing) process(es) that satisfies the minimum condition (cf. Definition 4.11). Moreover, both Cheridito et al. [11] and Soner et al. [87] explained the connection between Markovian 2BSDEs and a large class of fully nonlinear PDEs, which was one of the motivations that initiate this 2BSDEs topic.

Meanwhile, Peng [73, 75] independently introduced another framework (so-called G-framework) of a time consistent nonlinear expectation $E_G[\cdot]$, in which a new type of Brownian motion was constructed and the related Itô type stochastic calculus was established. By explicit constructions, Denis et al. [17] showed that G-expectation is in fact an upper expectation related to a non-dominated family $\mathcal{P}_G$ that consists of some probability measures similar to the elements in $\mathcal{P}_H$. In this regard, the G-framework is highly related to the 2BSDE one. Adopted the idea in Denis and Martini [19], Denis et al. [17] defined a Choquet capacity $\bar{C}(\cdot)$ on $(\Omega, \mathcal{B}(\Omega))$ as follows:

$$\bar{C}(A) := \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{P}(A),\ A \subset \mathcal{B}(\Omega),$$

and then they gave the notion of "quasi-surely" in a standard capacity-related vocabulary: a property holds true quasi-surely if and only if it holds outside a polar set, i.e., outside a set $A \in \mathcal{B}(Omega)$ that satisfies $\bar{C}(A) = 0$. We notice that this notion is a little bit stronger than the corresponding one in the 2BSDE framework, so that it yields another type of "quasi-sure" stochastic analysis. In this G-framework, Hu et al. [35] have worked on nonlinear BSDEs driven by G-brownian motion (GBSDEs), which is of the same form as (4.2) but can hold in the stronger "quasi-sure" sense. In that paper, the solution is an aggregated triple (Y, Z, K) which quasi-surely solves (4.2), where $-K$ is a decreasing G-martingale. To ensure that (4.2) is well defined in G-framework, an additional condition to the Lipschitz one is imposed on the regularity of the generator (cf. (H1) in Hu et al. [35]). This cost is intelligible since the definition of the G-Itô integrals is under a stronger norm induced by $E_G[\cdot]$ and it makes the space of admissible integrands smaller than the classical one.

Following the works of Soner et al. [88, 87, 86], Possamaï and Zhou [78] generalized the existence and uniqueness results for the 2BSDE whose generator has a quadratic growth. In order to make use of previous results of Tevzadze [93] for quadratic BSDEs, this work requires some additional condition, either on the terminal value or on the regularity of the generator. Our aim of this chapter is to remove these conditions, that is, to redo the job of Possamaï and Zhou [78] under some weaker assumptions of the type similar to that in Kobylanski [46] and Morlais [64].

In the classical framework, the quadratic BSDE is a powerful technique to deal with the utility maximization problems. El Karoui and Rouge [23] computed the value function of an exponential utility maximization problem when the strategies are confined to a convex cone, and they found that the optimal solution of its dual problem is related to a quadratic BSDE. In contrast to this, Hu et al. [34] and Morlais [64] directly treated the primal problem rather than the dual one and obtained a similar result without the convex condition on the constrain set. The value function was characterized also by a solution to a quadratic BSDE.

Corresponding to the work of Hu et al. [34], Matoussi et al. [62] found that a robust utility maximization problem with non-dominated models can be solved via the 2BSDE technique. This kind of problem was first consider by Denis and Kervarec [18] under a weakly compact class of probability measures. Because of this weakly compact assumption, one can find a least favorable probability in this class and work under this probability to find an optimal strategy similarly to how we solve the classical problem under a single probability. Matoussi et al. [62] characterized the value function by using a solution to a 2BSDE. However, the result in Matoussi et al. [62] has some limitations: for example, when the utility function is exponential, they are able to solve only the case that ξ is small enough or the border of the constraint domain satisfies an extra regularity condition. This limitations is due to the results of quadratic 2BSDEs in Possamaï and Zhou [78]. Since we shall remove these extra conditions adopted by Possamaï and Zhou [78], we can have a better result on this robust utility maximization problem.

This chapter is organized as follows: Section 4.2 includes preliminaries in 2BSDE theory; Section 4.3 introduces a representation theorem, a priori estimates and the uniqueness result for 2BSDEs with quadratic growth; Section 4.4 studies the existence of solutions while Section 4.5 considers the applications of quadratic 2BSDEs to robust maximization problems.

4.2 Preliminaries

The aim of this section is to give some basic definitions in 2BSDE theory introduced by Soner et al. [88, 87, 86] and Possamaï and Zhou [78]. The reader interested in a more detailed description of these notation is referred to these papers listed above.

4.2.1 The class of probability measures

Let $\Omega := \{\omega : \omega \in \mathcal{C}([0,1], \mathbb{R}^d), \omega_0 = 0\}$ be the canonical space equipped with the uniform norm $||\omega||_1^\infty := \sup_{0 \le t \le 1} |\omega_t|$, B the canonical process, $\mathcal{F}$ the filtration generated by B, $\mathcal{F}^+$ the right limit of $\mathcal{F}$.

We call $\mathbb{P}$ a local martingale measure if under which the canonical process B is a local martingale. By Karandikar [42], the quadratic variation process of B and its density can be defined universally, such that under each local martingale measure $\mathbb{P}$:

$$\langle B \rangle_t := B_t^2 - 2\int_0^t B_s dB_s \text{ and } \hat{a}_t := \overline{\lim_{\varepsilon \downarrow 0}} \frac{1}{\varepsilon}(\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}),\ 0 \le t \le 1,\ \mathbb{P} - a.s..$$

Adapting to Soner et al. [86], we denote $\overline{\mathcal{P}}_W$ the collection of all local martingale measures $\mathbb{P}$ such that $\langle B \rangle_t$ is absolutely continuous in t and $\hat{a}$ takes values in $\mathbb{S}_d^{>0}$, $\mathbb{P}$-a.s.. It is easy to verify that the following stochastic integral defines a $\mathbb{P}$-Brownian motion:

$$W_t^{\mathbb{P}} := \int_0^t \hat{a}_s^{-1/2} dB_s,\ 0 \le t \le 1.$$

We define a subclass of $\overline{\mathcal{P}}_W$ that consists of the probability measures induced by the strong formulation (cf. Lemma 8.1 in Soner et al. [86]):

$$\overline{\mathcal{P}}_S := \{\mathbb{P} \in \overline{\mathcal{P}}_W : \overline{\mathcal{F}^{W^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathcal{F}}^{\mathbb{P}}\},$$

where $\overline{\mathcal{F}}^{\mathbb{P}}$ ($\overline{\mathcal{F}^{W^{\mathbb{P}}}}^{\mathbb{P}}$, respectively) is the $\mathbb{P}$-augmentation of the filtration generated by B ($W^{\mathbb{P}}$, respectively).

4.2.2 The nonlinear generator

We consider a mapping $H_t(\omega, y, z, \eta) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$ and its Fenchel-Legendre conjugate with respect to η:

$$F_t(\omega, y, z, a) := \sup_{\eta \in D_H} \left\{ \frac{1}{2}\mathrm{tr}(a\eta) - H_t(\omega, y, z, \eta) \right\},\ a \in \mathbb{S}_d^{>0}.$$

where $D_H \subset \mathbb{R}^{d \times d}$ is a given subset that contains 0. For simplicity of notation, we note

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \text{ and } F_t^0 := \hat{F}_t(0, 0),$$

and we denote by $D_{F_t(y,z)}$ the domain of F in a for a fixed (t, ω, y, z). In accordance with settings in previous literature, we assume the following assumptions on F, which is needed for the "quasi-sure" technique:

(A1) *$D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z);*

(A2) *F is $\mathcal{F}$-progressively measurable and uniformly continuous in ω.*

4.2.3 The spaces and the norms

For the wellposedness of 2BSDEs, we consider a restrictive subclass $\mathcal{P}_H \subset \mathcal{P}_S$ defined as follows:

Definition 4.1 *Let $\mathcal{P}_H$ denote the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a}^{\mathbb{P}} \le \hat{a}_t \le \overline{a}^{\mathbb{P}}\ (\textit{usual partial ordering on } \mathbb{S}_d^{>0}) \textit{ and } \hat{a}_t \in D_{F_t},\ \lambda \times \mathbb{P} - a.e.,$$

for some $\underline{a}^{\mathbb{P}}$, $\overline{a}^{\mathbb{P}} \in \mathbb{S}_d^{>0}$ and all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$.

Remark 4.2 *Soner et al. [87] mentioned that the bounds $\underline{a}^{\mathbb{P}}$ and $\overline{a}^{\mathbb{P}}$ may vary in $\mathbb{P}$. Thanks to the quadratic growth assumption on F, i.e., (A3) in the sequel, $\hat{F}_t^0$ is bounded so that $\mathcal{P}_H$ is not empty in our case (cf. Remark 2.5 in Possamaï and Zhou [78]).*

Definition 4.3 *We say that a property holds $\mathcal{P}_H$-quasi-surely ($\mathcal{P}_H$-q.s.) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$.*

For each $p \geq 1$, L^p_H denotes the space of all $\mathcal{F}_1$-measurable scalar valued random variable ξ that satisfies

$$||\xi||_{L^p_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} E^{\mathbb{P}}[|\xi|^p] < +\infty.$$

Letting $p = +\infty$, we denote by L^∞_H the space of all $\mathbb{P}_H$-q.s. bounded random variable ξ with

$$||\xi||_{L^\infty_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} ||\xi||_{L^\infty(\mathbb{P})} < +\infty.$$

Let $\mathbb{D}^\infty_H$ denote the space of all $\mathbb{R}$-valued $\mathcal{F}^+$-progressively measurable process Y that satisfies

$$\mathcal{P}_H - q.s.\ c\grave{a}dl\grave{a}g \text{ and } ||Y||_{D^\infty_H} := \sup_{0\leq t\leq 1} ||Y_t||_{L^\infty_H} < +\infty,$$

and let $\mathbb{H}^2_H$ denote the space of all $\mathbb{R}^d$-valued $\mathcal{F}^+$-progressively measurable process Z that satisfies

$$||Z||^2_{\mathbb{H}^2_H} := \sup_{\mathbb{P}\in\mathcal{P}_H} E^{\mathbb{P}}\left[\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\right] < +\infty.$$

Remark 4.4 *We emphasize that the monotone convergence theorem no long holds true on each space listed above in this framework, i.e., that the monotone $\mathcal{P}_H$-q.s. convergence yields the convergence in norm may fail. As stated in Section 4 of Possamaï and Zhou [78], this is one of the main difficulties to prove the existence of quadratic 2BSDEs by a global approximation.*

With a little abuse of notation, we introduce the notion of $BMO(\mathcal{P}_H)$-martingale and its generator, which is an extension of the classical one. For the convenience of notation, H can refer to either a single process or a family of non-aggregated processes $\{H^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_H}$ in the definition and lemmas below.

Definition 4.5 *We call H a $BMO(\mathcal{P}_H)$-martingale if for each $\mathbb{P} \in \mathcal{P}_H$, $H^{\mathbb{P}}$ is a $\mathbb{P}$-square integrable martingale and*

$$||H||^2_{BMO_2(\mathcal{P}_H)} := \sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{\tau\in\mathcal{T}_0^1} ||E^{\mathbb{P}}_\tau[\langle H^{\mathbb{P}}\rangle_1 - \langle H^{\mathbb{P}}\rangle_\tau]||_{L^\infty(\mathbb{P})} < +\infty,$$

where $\mathcal{T}_0^1$ is the collection of all $\mathcal{F}$-stopping times τ that take values in $[0,1]$.

From the definition above, for a fixed $BMO(\mathcal{P}_H)$-martingale H, there exists a uniform constant $M_H > 0$, such that for all $\mathbb{P} \in \mathcal{P}_H$ and $\sigma \in \mathcal{T}_0^1$,

$$||H_{\cdot\wedge\sigma}||^2_{BMO_2(\mathbb{P})} \leq ||H||^2_{BMO_2(\mathbb{P})} \leq M_H.$$

Applying Theorem 2.4 and Theorem 3.1 in Kazamaki [43] under each $\mathbb{P} \in \mathcal{P}_H$, we have the following lemmas:

Lemma 4.6 *Suppose H is a $BMO(\mathcal{P}_H)$-martingale, then there exist two constants $r > 1$ and $C > 0$, such that*

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{0\leq t\leq 1} E^{\mathbb{P}}[|\mathcal{E}(H^{\mathbb{P}})_t|^r] \leq C,$$

and for some $q > 1$, the following reverse Hölder inequality holds under each $\mathbb{P} \in \mathcal{P}_H$ with a uniform constant C_{RH}: for each $0 \leq t_1 \leq t_2 \leq 1$,

$$E^{\mathbb{P}}_{t_1}[\mathcal{E}(H^{\mathbb{P}})^q_{t_2}] \leq C_{RH}\mathcal{E}(H^{\mathbb{P}})^q_{t_1},\ \mathbb{P} - a.s.,$$

where r, C, q and C_{RH} depend only on M_H.

Lemma 4.7 *Suppose H is a $BMO(\mathcal{P}_H)$-martingale, then there exist a $p > 1$ and a $C_E > 0$ that depend only on M_H, such that for each $t \in [0,1]$,*

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{\tau\in\mathcal{T}_0^t} \left|\left|E^{\mathbb{P}}_\tau\left[\left(\frac{\mathcal{E}(H^{\mathbb{P}})_\tau}{\mathcal{E}(H^{\mathbb{P}})_t}\right)^{\frac{1}{p-1}}\right]\right|\right|_{L^\infty(\mathbb{P})} \leq C_E.$$

Definition 4.8 *We call $Z \in \mathbb{H}^2_H$ a $BMO(\mathcal{P}_H)$-martingale generator if*

$$\begin{aligned}||Z||^2_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}} &:= \sup_{\mathbb{P}\in\mathcal{P}_H} \left|\left| \int_0^\cdot Z_t dB_t \right|\right|^2_{BMO_2(\mathbb{P})} \\ &= \sup_{\mathbb{P}\in\mathcal{P}_H} \sup_{\tau\in\mathcal{T}_0^1} \left|\left| E^{\mathbb{P}}_\tau \left[\int_\tau^1 |\hat{a}_t^{1/2} Z_t|^2 dt \right] \right|\right|_{L^\infty(\mathbb{P})} < +\infty.\end{aligned}$$

It is evident that if Z is a $BMO(\mathcal{P}_H)$-martingale generator, defining for each $\mathbb{P} \in \mathcal{P}_H$,

$$H^{\mathbb{P}}_t := \int_0^t Z_s dB_s, \ 0 \leq t \leq 1,$$

then H is a $BMO(\mathcal{P}_H)$-martingale. We denote by $\mathbb{H}^2_{BMO(\mathcal{P}_H)}$ the space of all $BMO(\mathcal{P}_H)$-martingale generators.

Applying energy inequality under each $\mathbb{P} \in \mathcal{P}_H$, we have the following lemma:

Lemma 4.9 *Suppose $Z \in \mathbb{H}^2_{BMO(\mathcal{P}_H)}$, then for each $p \geq 1$, $\mathbb{P} \in \mathcal{P}_H$ and all $\tau \in \mathcal{T}_0^1$,*

$$E^{\mathbb{P}}_\tau \left[\left(\int_\tau^1 |\hat{a}_t^{1/2} Z_t|^2 dt \right)^p \right] \leq C_p ||Z||^{2p}_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}}, \ \mathbb{P} - a.s..$$

Finally, we denote by $UC_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ and denote by $\mathcal{L}^\infty_H$ the closure of $UC_b(\Omega)$ under the norm $|| \cdot ||_{L^\infty_H}$.

4.2.4 Formulation to quadratic second order BSDEs

We shall consider the 2BSDE of the following form, which is first introduce in Soner et al. [87]:

$$Y_t = \xi + \int_t^1 \hat{F}_s(Y_s, Z_s) ds - \int_t^1 Z_s dB_s + K_1 - K_t, \ 0 \leq t \leq 1, \ \mathcal{P}_H - q.s.. \tag{4.3}$$

In addition to (A1)-(A2), we assume the following conditions on the generator F:

(A3) *F is continuous in (y, z) and has a quadratic growth, i.e., there exists a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, such that for all $(\omega, t, y, z, a) \in \Omega \times [0, 1] \times \mathbb{R} \times \mathbb{R}^d \times D_{F_t}$,*

$$|F_t(\omega, y, z, a)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|a^{1/2}z|^2; \tag{4.4}$$

(A4) *F is uniformly Lipschitz in y, i.e., there exists a $\mu > 0$, such that for all $(\omega, t, y, y', z, a) \in \Omega \times [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times D_{F_t}$,*

$$|F_t(\omega, y, z, a) - F_t(\omega, y', z, a)| \leq \mu|y - y'|;$$

(A5) *F is local Lipschitz in z, i.e., for each $(\omega, t, y, z, z', a) \in \Omega \times [0, 1] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times D_{F_t}$,*

$$|F_t(\omega, y, z, a) - F_t(\omega, y, z', a)| \leq C(1 + |a^{1/2}z| + |a^{1/2}z'|)|a^{1/2}(z - z')|.$$

Remark 4.10 *We have some comments on these conditions above: (A3) is a quadratic growth condition, which is similar to the condition for quadratic BSDEs studied by Kobylanski [46] and Marlais [64]; from (A4) and (A5) we can deduce some a priori estimate, and in the classical framework, analogous conditions are necessary for the proof of uniqueness in the articles of Hu et al. [34] and Morlais [64] for quadratic BSDEs. All these conditions above could be slightly weakened and further discussion will be made in Remark 4.34.*

Definition 4.11 *We say that $(Y, Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (4.3) if:*

- $Y_T = \xi$, $\mathcal{P}_H$-q.s.;

- The process $K^{\mathbb{P}}$ defined as below: for each $\mathbb{P} \in \mathcal{P}_H$,

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \le t \le 1,\ \mathbb{P} - a.s., \tag{4.5}$$

has increasing paths $\mathbb{P}$-a.s.;

- The family $\{K^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_H}$ satisfies the minimum condition: for each $\mathbb{P} \in \mathcal{P}_H$,

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^{\mathbb{P}}} E_t^{\mathbb{P}'}[K_T^{\mathbb{P}'}],\ 0 \le t \le 1,\ \mathbb{P} - a.s., \tag{4.6}$$

where

$$\mathcal{P}_H(t_1^+, \mathbb{P}) := \{\mathbb{P}' \in \mathcal{P}_H : \mathbb{P}'|_{\mathcal{F}_{t_1}^+} = \mathbb{P}|_{\mathcal{F}_{t_1}^+}\}.$$

Moreover, if the family $\{K^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_H}$ can be aggregated into a universal process K, we call (Y, Z, K) a solution of 2BSDE (4.3).

In the sequel, positive constants C and M will vary from line to line.

4.3 Representation and uniqueness of solutions to second order BSDEs

In this section, we give a representation theorem of solutions to the 2BSDE (4.3) under (A1)-(A5), which is similar to those in Soner et al. [87] and Possamaï and Zhou [78]. The representation theorem shows the relationship between the solution to the 2BSDE (4.3) and those to quadratic BSDEs with the generator $\hat{F}$ under each $\mathbb{P} \in \mathcal{P}_H$. Also, some a priori estimates to solutions is given which are useful to the proof of the existence.

4.3.1 Representation theorem

Before proceeding the argument, we first introduce a lemma (cf. Lemma 3.1 in Possamaï and Zhou [78]), the parallel version of which for quadratic BSDEs plays a very important role to show the connection between the boundedness of Y and the BMO property of the martingale part $\int ZdB$.

Lemma 4.12 *We assume (A1)-(A3) and $\xi \in L_H^\infty$. If $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ is a solution of the 2BSDE (4.3), then $Z \in \mathbb{H}^2_{BMO(\mathcal{P}_H)}$ and*

$$||Z||^2_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}} \le \frac{1}{\gamma^2} e^{4\gamma||Y||_{\mathbb{D}_H^\infty}} (1 + 2\gamma(\alpha + \beta||Y||_{\mathbb{D}_H^\infty})). \tag{4.7}$$

Consider the following quadratic BSDE under each $\mathbb{P} \in \mathcal{P}_H$:

$$y_s^{\mathbb{P}} = \eta + \int_s^t \hat{F}_u(y_u^{\mathbb{P}}, z_u^{\mathbb{P}})du - \int_s^t z_u^{\mathbb{P}} dB_u,\ 0 \le s \le t,\ \mathbb{P} - a.s., \tag{4.8}$$

where $0 \le s \le t \le 1$ and η is an $\mathcal{F}_t$-measurable random variable in $L^\infty(\mathbb{P})$. Under (A1)-(A5), the BSDE (4.8) admits a unique solution $(y^{\mathbb{P}}(t, \eta), z^{\mathbb{P}}(t, \eta))$ according to Kobylanski [46] and Morlais [64].

Then, we have the following representation theorem for the solution to the 2BSDE (4.3):

Theorem 4.13 *Let (A1)-(A5) hold. Assume that $\xi \in L_H^\infty$ and $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ is a solution to the 2BSDE (4.3). Then, for each $\mathbb{P} \in \mathcal{P}_H$ and all $0 \le t_1 \le t_2 \le 1$,*

$$Y_{t_1} = \underset{\mathbb{P}'\in\mathcal{P}_H(t_1^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.. \tag{4.9}$$

Remark 4.14 *Applying Theorem 2.7 (comparison principle) in Morlais [64], the theorem above also implies a comparison principle for quadratic 2BSDEs.*

Proof of Theorem 4.13: First of all, Lemma 4.12 shows that Z is a $BMO(\mathcal{P}_H)$-martingale generator, then we deduce by the BDG type inequalities, Lemma 4.9 and (4.7) that for each $p \geq 1$, $\mathbb{P} \in \mathcal{P}_H$ and all $0 \leq t_1 \leq t_2 \leq 1$,

$$E_{t_1}^{\mathbb{P}}[(K_{t_2}^{\mathbb{P}} - K_{t_1}^{\mathbb{P}})^p] \leq C_p := C(1 + e^{4p\gamma||Y||_{\mathbb{D}_H^\infty}})(1 + ||Y||_{\mathbb{D}_H^\infty}^p), \ \mathbb{P} - a.s.. \tag{4.10}$$

Since $\mathbb{P}$ is arbitrary in (4.10), we have

$$\underset{\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})}{\text{ess sup}^{\mathbb{P}}} \ E_{t_1}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^p] < C_p, \ \mathbb{P} - a.s..$$

We are now ready to prove that for a fixed $\mathbb{P} \in \mathcal{P}_H$ and all $0 \leq t_1 \leq t_2 \leq 1$,

$$Y_{t_1} \leq \underset{\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})}{\text{ess sup}^{\mathbb{P}}} \ y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \ \mathbb{P} - a.s.. \tag{4.11}$$

Fixing $t_2 \in [0,1]$, for each $\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})$, we note

$$\delta Y^{\mathbb{P}'} := Y - y^{\mathbb{P}'}(t_2, Y_{t_2}) \text{ and } \delta Z^{\mathbb{P}'} := Z - z^{\mathbb{P}'}(t_2, Y_{t_2}),$$

then, for each $t \in [0, t_2]$,

$$\delta Y_t^{\mathbb{P}'} = \int_t^{t_2} \lambda_s \delta Y_s ds - \int_t^{t_2} \delta Z_s \hat{a}_s^{1/2}(-\kappa_s^{\mathbb{P}'} ds + dW_s^{\mathbb{P}'}) + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'}, \ \mathbb{P}' - a.s.,$$

where λ is a scalar valued process and κ is an $\mathbb{R}^d$-valued process defined by

$$\kappa_t^{\mathbb{P}'} = \begin{cases} \frac{(\hat{F}_t(y_t^{\mathbb{P}'}(t_2,Y_{t_2}), z_t^{\mathbb{P}'}(t_2,Y_{t_2})) - \hat{F}_t(y_t^{\mathbb{P}'}(t_2,Y_{t_2}), Z_t))\hat{a}_t^{1/2}\delta Z_t}{|\hat{a}_t^{1/2}\delta Z_t|^2} & , \quad |\hat{a}_t^{1/2}\delta Z_t| \neq 0; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

By (A4) and (A5), we have $||\lambda||_{D^\infty(\mathbb{P}')} \leq \mu$ and κ satisfies

$$|\kappa_t^{\mathbb{P}'}| \leq 1 + |\hat{a}_t^{1/2} z_t^{\mathbb{P}'}(t_2, Y_{t_2})| + |\hat{a}_t^{1/2} Z_t|, \ 0 \leq t \leq t_2.$$

Defining

$$H_t^{\mathbb{P}'} := \int_0^t \kappa_s^{\mathbb{P}'} dW_s^{\mathbb{P}'}, \ 0 \leq t \leq t_2,$$

we have

$$||H^{\mathbb{P}'}||_{BMO(\mathbb{P}')}^2 \leq C(1 + ||\hat{a}^{1/2} z^{\mathbb{P}'}(t_2, Y_{t_2})||_{\mathbb{H}^2_{BMO(\mathbb{P}')}}^2 + ||\hat{a}^{1/2} Z||_{\mathbb{H}^2_{BMO(\mathbb{P}')}}^2). \tag{4.12}$$

Applying a priori estimates for quadratic BSDEs (cf. Lemma 3.1 in Morlais [64]), it is readily observed that

$$||\hat{a}^{1/2} z^{\mathbb{P}'}(t_2, Y_{t_2})||_{H^2_{BMO(\mathbb{P}')}}^2 \leq Ce^{4\gamma||Y||_{\mathbb{D}_H^\infty}}(1 + ||Y||_{\mathbb{D}_H^\infty}). \tag{4.13}$$

Putting (4.7) and (4.13) into (4.12), we deduce the following estimate uniformly in $\mathbb{P}'$:

$$||H^{\mathbb{P}'}||_{BMO(\mathbb{P}')}^2 \leq M_H, \tag{4.14}$$

where M_H depends only on $||Y||_{\mathbb{D}_H^\infty}$. This implies that H is a $BMO(\mathcal{P}_H)$-martingale.

Define a probability measure $\mathbb{Q}' \ll \mathbb{P}'$ by $\frac{d\mathbb{Q}'}{d\mathbb{P}'}|_{\mathcal{F}_t} = \mathcal{E}(\int_0^\cdot \kappa_s^{\mathbb{P}'} dW_s^{\mathbb{P}'})_t$ and a process $M_t := \exp(\int_{t_1}^t \lambda_s ds)$, $t_1 \leq t \leq t_2$. Applying Itô's formula to $M\delta Y$ under $\mathbb{Q}'$, we have

$$\begin{aligned} \delta Y_{t_1}^{\mathbb{P}'} = E_{t_1}^{\mathbb{Q}'}\left[\int_{t_1}^{t_2} M_t dK_t^{\mathbb{P}'}\right] &\leq E_{t_1}^{\mathbb{Q}'}[\sup_{t_1 \leq t \leq t_2}(M_t)(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})] \\ &\leq e^\mu E_{t_1}^{\mathbb{P}'}\left[\frac{\mathcal{E}(H^{\mathbb{P}'})_{t_2}}{\mathcal{E}(H^{\mathbb{P}'})_{t_1}}(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})\right], \ \mathbb{P} - a.s.. \end{aligned} \tag{4.15}$$

Thanks to (4.14) and Lemma 4.6, we can find uniformly for all $\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})$ two constants $q > 1$ and $C_{RH} > 0$, such that

$$\begin{aligned}
\delta Y_{t_1}^{\mathbb{P}'} &\leq C_{RH}^{1/q} e^{\mu} E_{t_1}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^p]^{1/p} \\
&\leq C_{RH}^{1/q} e^{\mu} (E_{t_1}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^{2p-1}])^{1/2p} E_{t_1}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})]^{1/2p} \\
&\leq C_{RH}^{1/q} e^{\mu} (\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})}^{\mathbb{P}} E_{t_1}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^{2p-1}])^{1/2p} E_{t_1}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})]^{1/2p} \\
&\leq C_{RH}^{1/q} C_{2p-1}^{1/2p} e^{\mu} E_{t_1}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})]^{1/2p}, \ \mathbb{P} - a.s.,
\end{aligned}$$

where $1/p + 1/q = 1$. Since C_{RH} and C_{2p-1} are independent of $\mathbb{P}'$, we can take essential infimum over all $\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})$ on the left-hand side of the inequality above and deduce by the minimum condition (4.6) that

$$\begin{aligned}
&Y_{t_1} - \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})}^{\mathbb{P}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}) \\
&\quad \leq C_{RH}^{1/q} C_{2p-1}^{1/2p} e^{\mu} \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_{t_1}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})]^{1/2p} = 0, \ \mathbb{P} - a.s..
\end{aligned}$$

From (4.15), it is easily observed that $\delta Y_{t_1}^{\mathbb{P}'} \geq 0$, $\mathbb{Q}'$-a.s. and thus, $\mathbb{P} - a.s.$, for all $\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})$, which directly yields the reverse inequality of (4.11). The proof of (4.9) is complete. □

4.3.2 A priori estimates

We now give some a priori estimates for quadratic 2BSDEs:

Lemma 4.15 *Let (A1)-(A5) hold. Assume that $\xi \in L_H^\infty$ and that $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ is a solution to 2BSDE (4.3). Then, there exists a $C > 0$ such that*

$$||Y||_{\mathbb{D}_H^\infty} \leq C(1 + ||\xi||_{L_H^\infty}) \text{ and } ||Z||^2_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}} \leq Ce^{4\gamma||\xi||_{L_H^\infty}}(1 + ||\xi||_{L_H^\infty}). \tag{4.16}$$

Proof: From (4.10) and a priori estimates for quadratic BSDEs, we deduce the left-hand side of (4.16), whereas the right-hand side comes after (4.7). □

Lemma 4.16 *Let (A1)-(A5) hold. Assume that $\xi^i \in L_H^\infty$ and that $(Y^i, Z^i) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$, $i = 1, 2$, are two solution to 2BSDE (4.3) corresponding to the two terminal values. Denote*

$$\begin{aligned}
&\delta\xi := \xi^1 - \xi^2, \ \delta Y := Y^1 - Y^2, \ \delta Z := Z^1 - Z^2, \\
&\delta K^{\mathbb{P}} := (K^1)^{\mathbb{P}} - (K^2)^{\mathbb{P}} \text{ and } \Delta\delta Y_t = \delta Y_t - \delta Y_{t^-},
\end{aligned}$$

then we have the following estimates

$$||\delta Y||_{\mathbb{D}_H^\infty} \leq C||\delta\xi||_{L_H^\infty}, \ ||\delta Z||_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}} \leq C(||\xi^1||_{L_H^\infty}, ||\xi^2||_{L_H^\infty})(||\delta\xi||_{L_H^\infty} + ||\delta\xi||^2_{L_H^\infty}),$$

and for a fixed $p > 0$, $t \in [0, 1]$ and each $\mathbb{P} \in \mathcal{P}_H$,

$$E_t^{\mathbb{P}}[\sup_{t \leq s \leq 1} |\delta K_s^{\mathbb{P}}|^p] \leq C(p, ||\xi^1||_{L_H^\infty}, ||\xi^2||_{L_H^\infty})(||\delta\xi||^{p/2}_{L_H^\infty} + ||\delta\xi||^p_{L_H^\infty}), \ \mathbb{P} - a.s..$$

Proof: Similarly to (4.15), we can easily obtain the first inequality. For the second one, we apply Itô's formula to δY^2, then we have for a fixed $\mathbb{P} \in \mathcal{P}_H$ and a $\tau \in \mathcal{T}_0^1$,

$$\begin{aligned}
|\delta Y_\tau|^2 + \int_\tau^1 |\hat{a}_t^{1/2} \delta Z_t|^2 dt &\leq |\delta\xi|^2 + 2\int_\tau^1 \delta Y_t (\hat{F}_t(Y_t^1, Z_t^1) - \hat{F}_t(Y_t^2, Z_t^2)) dt \\
&\quad - 2\int_\tau^1 \delta Y_t \delta Z_t dB_t + 2\int_\tau^1 \delta Y_{t^-} \delta dK_t^{\mathbb{P}} - \sum_{\tau < t \leq 1} |\Delta\delta Y_t|^2, \ \mathbb{P} - a.s..
\end{aligned}$$

Taking expectation on both sides and by (A3) and (4.10), we deduce

$$
\begin{aligned}
E_\tau^{\mathbb{P}}[\int_\tau^1 |\hat{a}_t^{1/2}\delta Z_t|^2 dt] &\leq ||\delta\xi||^2_{L_H^\infty} \\
&+ 2||\delta Y||_{\mathbb{D}_H^\infty}(2\alpha + \beta\sum_{i=1}^2 ||Y^i||_{\mathbb{D}_H^\infty} + \frac{\gamma}{2}\sum_{i=1}^2 ||Z^i||^2_{H^2_{BMO(\mathcal{P}_H)}}) \\
&+ 2||\delta Y||_{\mathbb{D}_H^\infty}(E_\tau^{\mathbb{P}}[(K^1)_1^{\mathbb{P}} - (K^1)_\tau^{\mathbb{P}}] + E_\tau^{\mathbb{P}}[(K^2)_1^{\mathbb{P}} - (K^2)_\tau^{\mathbb{P}}]) \\
&\leq C(||\xi^1||_{L_H^\infty}, ||\xi^2||_{L_H^\infty})(||\delta\xi||_{L_H^\infty} + ||\delta\xi||^2_{L_H^\infty}).
\end{aligned}
$$

For a fixed $p > 0$ and each $\mathbb{P} \in \mathcal{P}_H$, from (4.5), we have

$$
\begin{aligned}
E_t^{\mathbb{P}}[\sup_{t\leq s\leq 1} |\delta K_s^{\mathbb{P}}|^p] &\leq C_p\Bigg(||\delta\xi||^p_{L_H^\infty} + E_t^{\mathbb{P}}\Bigg[\Bigg(\int_t^1 (1 + |\hat{a}_s^{1/2} Z_s^1| + |\hat{a}_s^{1/2} Z_s^2|)^2 ds\Bigg)^p\Bigg]^{1/2} \\
&\times E_t^{\mathbb{P}}\Bigg[\Bigg(\int_t^1 |\hat{a}_s^{1/2}\delta Z_s|^2 ds\Bigg)^p\Bigg]^{1/2} + E_t^{\mathbb{P}}\Bigg[\int_t^1 |\hat{a}_s^{1/2}\delta Z_s|^2 ds\Bigg]^{p/2}\Bigg) \\
&\leq C_p\Bigg(||\delta\xi||^p_{L_H^\infty} + ||\delta Z||^{p/2}_{\mathbb{H}^2_{BMO(\mathcal{P}_H)}}\Bigg(1 + \sum_{i=1}^2 ||Z^i||^p_{H^2_{BMO(\mathcal{P}_H)}}\Bigg)\Bigg) \\
&\leq C(p, ||\xi^1||_{L_H^\infty}, ||\xi^2||_{L_H^\infty})(||\delta\xi||^{p/2}_{L_H^\infty} + ||\delta\xi||^p_{L_H^\infty}).
\end{aligned}
$$

We complete the proof. □

By either Theorem 4.13 or Lemma 4.16, we deduce immediately the uniqueness of Y. We observe that $d\langle Y, B\rangle_t = Z_t d\langle B\rangle_t$, $0 \leq t \leq 1$, $\mathcal{P}_H$-q.s., which implies the uniqueness of Z.

4.4 Existence of solutions to second order BSDEs

In this section, we provide the existence result for the 2BSDE (4.3) under (A1)-(A5) by a pathwise construction introduced in Soner et al. [87, 88] with the so-called technique of regular conditional probability distribution (r.c.p.d.), which can be find in Stroock and Varadhan [91].

4.4.1 Regular conditional probability distributions

For the convenience of the reader, we recall some notations of r.c.p.d. in Soner et al. [88].

- For each $t \in [0,1]$, let $\Omega^t := \{\tilde{\omega} \in \mathcal{C}([t,1],\mathbb{R}^d), \tilde{\omega}(t) = 0\}$ be the shifted space, B^t the shifted canonical process, $\mathcal{F}^t$ the shifted filtration generated by B^t.
- For each $0 \leq s \leq t \leq 1$, $\omega \in \Omega^s$ and $\tilde{\omega} \in \Omega^t$, we define the concatenation path $\omega \otimes_t \tilde{\omega} \in \Omega^s$ by

$$(\omega \otimes_t \tilde{\omega})_u := \omega_u \mathbf{1}_{[s,t)}(u) + (\omega_t + \tilde{\omega}_u)\mathbf{1}_{[t,1]}(u),\ u \in [s,1].$$

- For each $0 \leq s \leq t \leq 1$, $\omega \in \Omega^s$ and an $\mathcal{F}_1^s$-measurable random variable ξ on Ω^s, we define the shifted $\mathcal{F}_1^t$-measurable random variable $\xi^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\tilde{\omega}) := \xi(\omega \otimes_t \tilde{\omega}),\ \tilde{\omega} \in \Omega^t.$$

- For each $0 \leq s \leq t \leq 1$, the shifted process $X^{t,\omega}$ of an $\mathcal{F}^s$-progressively measurable X is $\mathcal{F}^t$-progressively measurable.
- For each $t \in [0,1]$ and $\omega \in \Omega$, we define our shifted generator by

$$\hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) := F_s(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s^t(\tilde{\omega})),\ (s,\tilde{\omega}) \in [t,1] \times \Omega^t.$$

- For each $t \in [0,1]$, $\mathcal{P}_H^t$ denotes the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S^t$ such that

$$\underline{a}^{\mathbb{P}} \leq \hat{a}_s^t \leq \overline{a}^{\mathbb{P}} \text{ and } \hat{a}_s^t \in D_{F_s},\ \lambda \times \mathbb{P} - a.e.,$$

  for some $\underline{a}^{\mathbb{P}}$, $\overline{a}^{\mathbb{P}} \in \mathbb{S}_d^{>0}$ and all $(y,z) \in \mathbb{R} \times \mathbb{R}^d$.

- For each $(\omega, t) \in \Omega \times [0,1]$ and $\mathbb{P} \in \mathcal{P}_H$, the r.c.p.d $\mathbb{P}_t^\omega$ of $\mathbb{P}$ induces naturally a probability measure $\mathbb{P}^{t,\omega}$ on $(\Omega^t, \mathcal{F}_1^t)$ which satisfies that for each bounded and $\mathcal{F}_1$-measurable random variable ξ,

$$E^{\mathbb{P}_t^\omega}[\xi] = E^{\mathbb{P}^{t,\omega}}[\xi^{t,\omega}].$$

- By Lemma 4.1 in Soner et al. [88], $\mathbb{P}^{t,\omega}$ is an element in $\mathcal{P}_H^t$ and for each $t \in [0,1]$ and $\mathbb{P} \in \mathcal{P}_H$, it holds for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$F_s(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s^t(\tilde{\omega})) = F_s(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s(\omega \otimes_t \tilde{\omega})),\ \lambda \times \mathbb{P}^{t,\omega} - a.e..$$

4.4.2 Existence result

For a fixed $\omega \in \Omega$ and $0 \le t_1 \le t_2 \le 1$, we consider a quadratic BSDE of the following type on the shifted space Ω^{t_1} under each $\mathbb{P}^{t_1} \in \mathcal{P}_H^{t_1}$:

$$\begin{aligned} y_s^{\mathbb{P}^{t_1},t_1,\omega} = \eta^{t_1,\omega} &+ \int_s^{t_2} \hat{F}_u^{t_1,\omega}(y_u^{\mathbb{P}^{t_1},t_1,\omega}, z_u^{\mathbb{P}^{t_1},t_1,\omega})du \\ &- \int_s^{t_2} z_u^{\mathbb{P}^{t_1},t_1,\omega} dB_u^{t_1},\ t_1 \le s \le t_2,\ \mathbb{P}^{t_1} - a.s., \end{aligned} \tag{4.17}$$

where $\eta \in L_H^\infty$ is an $\mathcal{F}_{t_2}$-measurable random variable. It is well known that (4.17) admits a unique solution $(y^{\mathbb{P}^{t_1},t_1,\omega}(t_2,\eta), z^{\mathbb{P}^{t_1},t_1,\omega}(t_2,\eta))$ under (A1)-(A5). In view of the Blumenthal zero-one law, $y_{t_1}^{\mathbb{P}^{t_1},t_1,\omega}(t_2,\eta)$ is deterministic $\mathbb{P}^{t_1}$-a.s. for any given η and $\mathbb{P}^{t_1}$.

The following lemma describes the relationship between $y_t^{\mathbb{P}}(1,\xi)$ and $y_t^{\mathbb{P}^{t,\omega},t,\omega}(1,\xi)$, where the former is the solution of (4.8) with parameters $(1,\xi)$ under a fixed $\mathbb{P} \in \mathcal{P}_H$ and the latter is the solution of (4.17) when t takes the place of t_1, $\mathbb{P}^t$ is in fact the r.c.p.d $\mathbb{P}^{t,\omega}$ of $\mathbb{P}$ and $(t_2,\eta) = (1,\xi)$.

Lemma 4.17 *Assume (A1)-(A5) hold. For a given $\xi \in L_H^\infty$ and a fixed $\mathbb{P} \in \mathcal{P}_H$, we have, for each $t \in [0,1]$ and $\mathbb{P}$-a.s. $\omega \in \Omega$,*

$$y_t^{\mathbb{P}}(1,\xi)(\omega) = y_t^{\mathbb{P}^{t,\omega},t,\omega}(1,\xi). \tag{4.18}$$

Proof: Similar to the proof of Lemma 4.1 in Soner et al. [88], we have the following conclusion: because $\xi \in L^\infty(\mathbb{P})$, for $\mathbb{P}$-a.s. $\omega \in \Omega$, $|\xi^{t,\omega}| \le ||\xi||_{L^\infty(\mathbb{P})}$, $\mathbb{P}^{t,\omega}$-a.s.. Thus, (4.17) is well defined under our setting and the right-hand side of (4.18) is the unique solution to (4.17).

We emphasize that the wellposedness of both (4.8) and (4.17) as well as estimates of solutions are already provided by Kobylanski [46] and Morlais [64]. Our job here is only to redo the construction of two sequences formed by solutions of Lipschitz BSDEs, which approximate solutions on both sides of (4.18).

By Lemma 3.1 in Morlais [64], we can find a constant $M := e^\beta(\alpha + ||\xi(\omega)||_{L^\infty(\mathbb{P})})$, which is the bound of both sides of (4.18). Then, we choose a $\mathcal{C}^\infty(\mathbb{R})$ function which takes value in $[0,1]$ and satisfies that

$$\phi(u) = \begin{cases} 1 & , \quad u \in [e^{-\gamma M}, e^{\gamma M}]; \\ 0 & , \quad u \in (-\infty, e^{-\gamma(M+1)}] \cup [e^{\gamma(M+1)}, +\infty). \end{cases}$$

We can verify that for each $t \in [0,1]$,

$$\mathcal{Y}_t^{\mathbb{P}}(1, e^{\gamma\xi}, \hat{G}) := \exp(\gamma y_t^{\mathbb{P}}(1,\xi)), \tag{4.19}$$

solves a quadratic BSDE with parameters $(1, e^{\gamma\xi})$ and the generator $\hat{G}$ of the following form: for each $(\omega, t, \mathcal{Y}, \mathcal{Z}) \in \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R}^d$,

$$\hat{G}_t(\omega, \mathcal{Y}, \mathcal{Z}) := \phi(\mathcal{Y})\bigg(\gamma\mathcal{Y}\hat{F}_t\Big(\omega, \frac{\ln(\mathcal{Y})}{\gamma}, \frac{\mathcal{Z}}{\gamma\mathcal{Y}}\Big) - \frac{1}{2\mathcal{Y}}|\hat{a}_t^{1/2}(\omega)\mathcal{Z}|^2\bigg). \tag{4.20}$$

On the other hand, fixing $(\omega, t) \in \Omega \times [0,1]$,

$$\mathcal{Y}_s^{\mathbb{P}^{t,\omega},t,\omega}(1, e^{\gamma\xi}, \hat{G}^{t,\omega})(\tilde{\omega}) := \exp(\gamma y_s^{\mathbb{P}^{t,\omega},t,\omega}(1,\xi)(\tilde{\omega})),\ t \le s \le 1, \tag{4.21}$$

defines a solution that solves a quadratic BSDE under $\mathbb{P}^{t,\omega}$ with parameters $(1, e^{\gamma\xi})$ and the generator $\hat{G}^{t,\omega}$ of the following form: for each $(\tilde{\omega}, s, \mathcal{Y}, \mathcal{Z}) \in \Omega^t \times [t,1] \times \mathbb{R} \times \mathbb{R}^d$,

$$\hat{G}^{t,\omega}_s(\tilde{\omega}, \mathcal{Y}, \mathcal{Z}) := \phi(\mathcal{Y})\bigg(\gamma\mathcal{Y}\hat{F}^{t,\omega}_s\bigg(\tilde{\omega}, \frac{\ln(\mathcal{Y})}{\gamma}, \frac{\mathcal{Z}}{\gamma\mathcal{Y}}\bigg) - \frac{1}{2\mathcal{Y}}|(\hat{a}^t_s)^{1/2}(\tilde{\omega})\mathcal{Z}|^2\bigg).$$

Now, our main aim is changed into that for each $t \in [0,1]$ and $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$\mathcal{Y}^{\mathbb{P}}_t(1, e^{\gamma\xi}, \hat{G})(\omega) = \mathcal{Y}^{\mathbb{P}^{t,\omega},t,\omega}_t(1, e^{\gamma\xi}, \hat{G}^{t,\omega}).$$

For each $(\omega, t, \mathcal{Y}, \mathcal{Z}) \in \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R}^d$, we set

$$\hat{G}^n_t(\omega, \mathcal{Y}, \mathcal{Z}) := \sup_{(p,q)\in\mathbb{Q}\times\mathbb{Q}^d}\{\hat{G}_t(\omega, p, q) - n|(p-\mathcal{Y})| - n|\hat{a}^{1/2}_t(\omega)(q-\mathcal{Z})|\},\ n \in \mathbb{N}, \tag{4.22}$$

and also for fixed $(\omega, t) \in \Omega \times [0,1]$ and each $(\tilde{\omega}, s, \mathcal{Y}, \mathcal{Z}) \in \Omega^t \times [t,1] \times \mathbb{R} \times \mathbb{R}^d$, we define

$$\begin{aligned}&(\hat{G}^{t,\omega})^n_s(\tilde{\omega}, \mathcal{Y}, \mathcal{Z})\\ &\quad := \sup_{(p,q)\in\mathbb{Q}\times\mathbb{Q}^d}\{\hat{G}^{t,\omega}_s(\tilde{\omega}, p, q) - n|p-\mathcal{Y}| - n|(\hat{a}^t_s)^{1/2}(\tilde{\omega})(q-\mathcal{Z})|\},\ n \in \mathbb{N}.\end{aligned}$$

By Lemma 4.1 in Soner et al. [88], for each $t \in [0,1]$, $\mathbb{P}$-a.s. $\omega \in \Omega$ and each $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned}\hat{F}^{t,\omega}_s(\tilde{\omega}, y, z) &= F^{t,\omega}_s(\tilde{\omega}, y, z, \hat{a}^t_s(\tilde{\omega})) = F_s(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s(\omega \otimes_t \tilde{\omega}))\\ &= (\hat{F}(\cdot,\cdot))^{t,\omega}_s(\tilde{\omega}, y, z),\ \text{and}\ \hat{a}^t_s(\tilde{\omega}) = \hat{a}^{t,\omega}_s(\tilde{\omega}),\ \lambda \times \mathbb{P}^{t,\omega} - a.e..\end{aligned} \tag{4.23}$$

We call $(\hat{F}(\cdot,\cdot))^{t,\omega}$ the globally shifted generator of $\hat{F}$. From (4.23), we can deduce that for each $t \in [0,1]$, $\mathbb{P}$-a.s. $\omega \in \Omega$ and each $(\mathcal{Y}, \mathcal{Z}) \in \mathbb{R} \times \mathbb{R}^d$,

$$\hat{G}^{t,\omega}_s(\tilde{\omega}, \mathcal{Y}, \mathcal{Z}) = (\hat{G}(\cdot,\cdot))^{t,\omega}_s(\tilde{\omega}, \mathcal{Y}, \mathcal{Z}),\ \lambda \times \mathbb{P}^{t,\omega} - a.e.,$$

and furthermore that for each $n \in \mathbb{N}$,

$$(\hat{G}^{t,\omega})^n_s(\tilde{\omega}, \mathcal{Y}, \mathcal{Z}) = (\hat{G}^n(\cdot,\cdot))^{t,\omega}_s(\tilde{\omega}, \mathcal{Y}, \mathcal{Z}),\ \lambda \times \mathbb{P}^{t,\omega} - a.e.. \tag{4.24}$$

Moreover, it is easy to verify that for each $(\omega, t, \mathcal{Y}, \mathcal{Z}) \in \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned}-e^{M+1}(\alpha\gamma + \beta(M+1)) - e^{M+1}|\hat{a}^{1/2}_t(\tilde{\omega})\mathcal{Z}|^2 &\le \hat{G}_t(\omega, \mathcal{Y}, \mathcal{Z})\\ &\le \hat{G}^{n+1}_t(\omega, \mathcal{Y}, \mathcal{Z}) \le \hat{G}^n_t(\omega, \mathcal{Y}, \mathcal{Z}) \le e^{M+1}(\alpha\gamma + \beta(M+1)),\end{aligned}$$

and $\hat{G}^n_t(\omega, \mathcal{Y}, \mathcal{Z}) \downarrow \hat{G}_t(\omega, \mathcal{Y}, \mathcal{Z})$ uniformly on compact sets in $[0,1] \times \mathbb{R} \times \mathbb{R}^d$. Similarly, for fixed $(\omega, t) \in \Omega \times [0,1]$ and each $(\tilde{\omega}, s, \mathcal{Y}, \mathcal{Z}) \in \Omega^t \times [t,1] \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned}-e^{M+1}(\alpha\gamma + \beta(M+1)) - e^{M+1}|(\hat{a}^t_s)^{1/2}(\tilde{\omega})\mathcal{Z}|^2 &\le \hat{G}^{t,\omega}_s(\omega, \mathcal{Y}, \mathcal{Z})\\ &\le (\hat{G}^{t,\omega})^{n+1}_s(\omega, \mathcal{Y}, \mathcal{Z}) \le (\hat{G}^{t,\omega})^n_s(\omega, \mathcal{Y}, \mathcal{Z}) \le e^{M+1}(\alpha\gamma + \beta(M+1)),\end{aligned}$$

and $(\hat{G}^{t,\omega})^n_t(\omega, \mathcal{Y}, \mathcal{Z}) \downarrow (\hat{G}^{t,\omega})_t(\omega, \mathcal{Y}, \mathcal{Z})$ uniformly on compact sets in $[t,1] \times \mathbb{R} \times \mathbb{R}^d$.

By Lemma 3.3 (monotone stability) in Morlais [64], we have, for each $t \in [0,1]$ and $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$\mathcal{Y}^{\mathbb{P}}_t(1, e^{\gamma\xi}, \hat{G}^n)(\omega) \downarrow \mathcal{Y}^{\mathbb{P}}_t(1, e^{\gamma\xi}, \hat{G})(\omega),\ \text{as}\ n \to +\infty, \tag{4.25}$$

and for fixed $(\omega, t) \in \Omega \times [0,1]$,

$$\mathcal{Y}^{\mathbb{P}^{t,\omega},t,\omega}_t(1, e^{\gamma\xi}, (\hat{G}^{t,\omega})^n) \downarrow \mathcal{Y}^{\mathbb{P}^{t,\omega},t,\omega}_t(1, e^{\gamma\xi}, \hat{G}^{t,\omega}),\ \text{as}\ n \to +\infty. \tag{4.26}$$

To obtain the desired result, it suffices to prove that for each $n \in \mathbb{N}$, a fixed $t \in [0,1]$ and $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$\mathcal{Y}^{\mathbb{P}}_t(1, e^{\gamma\xi}, \hat{G}^n)(\omega) = \mathcal{Y}^{\mathbb{P}^{t,\omega},t,\omega}_t(1, e^{\gamma\xi}, (\hat{G}^{t,\omega})^n). \tag{4.27}$$

We notice that the generators of both sides of (4.27) satisfy the following uniform Lipschitz conditions: for each $n \in \mathbb{N}$ and $(\omega, t, \mathcal{Y}^1, \mathcal{Y}^2, \mathcal{Z}^1, \mathcal{Z}^2) \in \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$|\hat{G}_t^n(\omega, \mathcal{Y}^1, \mathcal{Z}^1) - \hat{G}_t^n(\omega, \mathcal{Y}^2, \mathcal{Z}^2)| \leq n|\mathcal{Y}^1 - \mathcal{Y}^2| + n|\hat{a}_t^{1/2}(\omega)(\mathcal{Z}^1 - \mathcal{Z}^2)|;$$

and for fixed $(\omega, t) \in \Omega \times [0,1]$, each $n \in \mathbb{N}$ and $(\tilde{\omega}, s, \mathcal{Y}^1, \mathcal{Y}^2, \mathcal{Z}^1, \mathcal{Z}^2) \in \Omega^t \times [t,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$|(\hat{G}^{t,\omega})_s^n(\tilde{\omega}, \mathcal{Y}^1, \mathcal{Z}^1) - (\hat{G}^{t,\omega})_s^n(\tilde{\omega}, \mathcal{Y}^2, \mathcal{Z}^2)| \leq n|\mathcal{Y}^1 - \mathcal{Y}^2| + n|(\hat{a}_s^t)^{1/2}(\tilde{\omega})(\mathcal{Z}^1 - \mathcal{Z}^2)|.$$

Since solutions of these Lipschitz BSDEs can be constructed via Picard iteration, from (4.24), we can obtain (4.27) (cf. (i) in the proof of Proposition 4.7 in Soner et al. [88] and (i) in the proof of Proposition 5.1 in Possamaï and Zhou [78]). Then, (4.25) and (4.26) give the desired result. ☐

Remark 4.18 *The lemma above is the key point of this chapter, which ensures us to prove the following proposition under the Kobylanski's [46] type condition instead of Tevzadze's [93] type one, which is adopted by Possamaï and Zhou [78] to make sure that solutions of quadratic BSDEs on both original and shifted spaces can be constructed via Picard iteration, so that the statement corresponding to (4.18) in Soner et al. [88] for Lipschitz BSDEs still holds. In this chapter, the lemma above is proved by a monotonic convergence technique for classical BSDEs under a fixed $\mathbb{P}$, but it is still difficult to obtain a globally monotonic convergence theorem for quadratic 2BSDEs, as Possamaï and Zhou [78] have already stated.*

Similarly to the one in Soner et al. [88], we define the following value process V_t pathwisely: for each $(\omega, t) \in \Omega \times [0,1]$,

$$V_t(\omega) := \sup_{\mathbb{P}^t \in \mathcal{P}_H^t} y_t^{\mathbb{P}^t,t,\omega}(1, \xi). \tag{4.28}$$

For the rest part of the proof of the existence, we assume moreover that the terminal value ξ is an element in $UC_b(\Omega)$. Therefore, it is readily observed that for all $(\omega, t) \in \Omega \times [0,1]$,

$$|V_t(\omega)| \leq C(1 + \sup_{\omega \in \Omega} |\xi(\omega)|), \tag{4.29}$$

and there exists a modulus of continuity ρ, such that for each $t \in [0,1]$ and $(\omega, \omega', \tilde{\omega}) \in \Omega \times \Omega \times \Omega^t$,

$$|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})| \leq \rho(||\omega - \omega'||_t^\infty).$$

Recalling the uniform continuity of F in ω, we have moreover that for each $0 \leq t \leq s \leq 1$, $(\omega, \omega', \tilde{\omega}, y, z) \in \Omega \times \Omega \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d$,

$$|\hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) - \hat{F}_s^{t,\omega'}(\tilde{\omega}, y, z)| \leq \rho(||\omega - \omega'||_t^\infty).$$

We define

$$\delta y := y^{\mathbb{P}^t,t,\omega}(1,\xi) - y^{\mathbb{P}^t,t,\omega'}(1,\xi),\ \delta z := z^{\mathbb{P}^t,t,\omega}(1,\xi) - z^{\mathbb{P}^t,t,\omega'}(1,\xi),$$
$$\delta\xi := \xi^{t,\omega} - \xi^{t,\omega'},\ \delta\hat{F}(y,z) := \hat{F}^{t,\omega}(y,z) - \hat{F}^{t,\omega'}(y,z).$$

Proceeding the same in the proof of Theorem 4.13, for each $(\omega, \omega', t) \in \Omega \times \Omega \times [0,1]$ and a fixed $\mathbb{P}^t \in \mathcal{P}_H^t$, we can find a $\mathbb{Q}^t \ll \mathbb{P}^t$ and a bounded process M, such that

$$|\delta y_t| = E^{\mathbb{Q}^t}\left[M_1 \delta\xi + \int_t^1 M_s \delta\hat{F}_s(y_s^{\mathbb{P}^t,t,\omega}(1,\xi), z_s^{\mathbb{P}^t,t,\omega}(1,\xi))ds\right] \leq C\rho(||\omega - \omega'||_t^\infty). \tag{4.30}$$

By the arbitrariness of $\mathbb{P}^t$, it follows that

$$|V_t(\omega) - V_t(\omega')| \leq C\rho(||\omega - \omega'||_t^\infty), \tag{4.31}$$

from which we can deduce that $V_t \in \mathcal{F}_t$.

Parallel to Proposition 4.7 in Soner et al. [88] and Proposition 5.1 in Possamaï and Zhou [78], we give the following dynamic programming principle:

Proposition 4.19 *Under (A1)-(A5) and for a given $\xi \in UC_b$, we have, for each $0 \le t_1 \le t_2 \le 1$ and $\omega \in \Omega$,*

$$V_{t_1}(\omega) = \sup_{\mathbb{P}^{t_1} \in \mathcal{P}_H^{t_1}} y_{t_1}^{\mathbb{P}^{t_1}, t_1, \omega}(t_2, V_{t_2}). \tag{4.32}$$

Proof: Without loss of generality, we only need to prove the case when $t_1 = 0$ and $t_2 = t$, i.e.,

$$V_0 = \sup_{\mathbb{P} \in \mathcal{P}_H} y_0^{\mathbb{P}}(t, V_t).$$

Fixing $\mathbb{P} \in \mathcal{P}_H$, for each $\omega \in \Omega$ and $t \in [0,1]$, $\mathbb{P}^{t,\omega} \in \mathcal{P}_H^t$. By Lemma 4.17 and from (4.28), we have, for each $t \in [0,1]$ and $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$y_t^{\mathbb{P}}(1,\xi)(\omega) = y_t^{\mathbb{P}^{t,\omega}, t, \omega}(1,\xi) \le \sup_{\mathbb{P}^t \in \mathcal{P}_H^t} y_t^{\mathbb{P}^t, t, \omega}(1,\xi) = V_t(\omega).$$

Applying Theorem 2.7 (comparison principle) in Morlais [64], it follows that $V_0 \le \sup_{\mathbb{P} \in \mathcal{P}_H} y_0^{\mathbb{P}}(t, V_t)$.

We omit the rest part of the proof, i.e., the proof of the reverse inequality of (4.32) via the r.c.p.d. technique, since it goes in the same way as the one in Soner et al. [88] and Possamaï and Zhou [78]. □

We shall turn to the Doob-Meyer type decomposition of V based on some results for quadratic g-supermartingales in Ma and Yao [59]. These results were obtained under the assumptions for the proof of uniqueness in Kobylanski [46], since these assumptions ensure that the wellposedness of the corresponding quadratic BSDEs, so that the g-expectation can be well defined. However, Morlais [64] also provided the wellposedness of quadratic BSDEs under (A3)-(A5) for BSDEs of the form (4.8), then the applicability of these arguments to such type of $\hat{F}$-supermartingales will not alter under each $\mathbb{P} \in \mathcal{P}_H$.

For a fixed $\mathbb{P} \in \mathcal{P}_H$, from (4.32) and by Lemma 4.17, we have for each $0 \le t_1 \le t_2 \le 1$,

$$V_{t_1} \ge y_{t_1}^{\mathbb{P}}(t_2, V_{t_2}),\ \mathbb{P} - a.s.. \tag{4.33}$$

Thus, by Definition 5.1 in Ma and Yao [59], V is an $\hat{F}$-supermartingale under $\mathbb{P}$. Then, for each $(\omega, t) \in \Omega \times [0,1]$, we define

$$V_t^+(\omega) := \limsup_{\mathbb{Q} \cap (t,1] \ni r \downarrow t} V_r(\omega).$$

Applying corollary 5.6 (downcrossing inequality) in Ma and Yao [59] , one can see that for $\mathbb{P}$-a.s. $\omega \in \Omega$, $\lim_{\mathbb{Q} \cap (t,1] \ni r \downarrow t} V_r$ exists for all $t \in [0,1]$. Therefore, we have

$$V_t^+ = \lim_{\mathbb{Q} \cap (t,1] \ni r \downarrow t} V_r,\ 0 \le t \le 1,\ \mathcal{P}_H - q.s., \tag{4.34}$$

which implies that V^+ has $\mathcal{P}_H$-q.s. càdlàg paths.

The following proposition (corresponding to Proposition 4.10 and 4.11 in Soner et al. [88] and Proposition 5.2 in Possamaï and Zhou [78]) demonstrates the relationship between V and V^+, from the second part of which, we can deduce that V is a càdlàg $\hat{F}$-supermartingale under each $\mathbb{P} \in \mathcal{P}_H$, then we apply the Doob-Meyer type decomposition theorem (cf. Theorem 5.8 in Ma and Yao [59]) directly to V.

Proposition 4.20 *Assume (A1)-(A5) hold. For a given $\xi \in UC_b(\Omega)$ and a fixed $\mathbb{P} \in \mathcal{P}_H$, we define*

$$V_t^{\mathbb{P}} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})} y_t^{\mathbb{P}'}(1,\xi) \text{ and } V_t^{\mathbb{P},+} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})} y_t^{\mathbb{P}'}(1,\xi). \tag{4.35}$$

Then, we have

$$V_t = V_t^{\mathbb{P}} \text{ and } V_t^+ = V_t^{\mathbb{P},+},\ 0 \le t \le 1,\ \mathbb{P} - a.s..$$

Moreover,

$$V_t = V_t^+,\ 0 \le t \le 1,\ \mathcal{P}_H - q.s.. \tag{4.36}$$

Proof: For the proof of the first equality in (4.35) and that $V_t^+ \geq V_t^{\mathbb{P},+}$, we can proceed in the same way as in the proof of Proposition 4.10 in Soner et al. [88]. Here, we would like only to prove that for a fixed $\mathbb{P} \in \mathcal{P}_H$,

$$V_t^+ \leq V_t^{\mathbb{P},+},\ 0 \leq t \leq 1,\ \mathbb{P}-a.s.,$$

since the technique will be a little different.

Fixing $\mathbb{P} \in \mathcal{P}_H$, $t \in [0,1]$ and $r \in \mathbb{Q} \cap (t,1]$, from the first equality, we have

$$V_r^{\mathbb{P}} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})} y_r^{\mathbb{P}'}(1,\xi).$$

Following Step 3 in the proof of Theorem 4.3 in Soner et al. [87], we could find a sequence of probability measures such that $\{\mathbb{P}_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_H(r,\mathbb{P}) \subset \mathcal{P}_H(t^+,\mathbb{P})$ and $y_r^{\mathbb{P}_n}(1,\xi) \uparrow V_r$, $\mathbb{P}$-a.s.. We consider the following BSDE with parameters (r, V_r) and the generator $\hat{F}$:

$$y_s^{\mathbb{P}} = V_r + \int_s^r \hat{F}_u(y_u^{\mathbb{P}}, z_u^{\mathbb{P}})du - \int_s^r z_u^{\mathbb{P}} dB_u,\ 0 \leq s \leq r,\ \mathbb{P}-a.s.,$$

and denote by $(y^{\mathbb{P}}(r,V_r), z^{\mathbb{P}}(r,V_r))$ its solution. Then, it follows by Lemma 3.3 (monotone stability) in Morlais [64] that

$$y_t^{\mathbb{P}}(r,V_r) = y_t^{\mathbb{P}}(r, \lim_{n\to+\infty} y_r^{\mathbb{P}_n}(1,\xi)) = \lim_{n\to+\infty} y_t^{\mathbb{P}^n}(1,\xi) \leq V_t^{\mathbb{P},+},\ \mathbb{P}-a.s..$$

Now, our aim is to find a sequence $\{r_m\}_{m\in\mathbb{N}} \subset (t,1]$ such that $r_m \downarrow t$ and

$$\lim_{m\to+\infty} y_t^{\mathbb{P}}(r_m, V_{r_m}) = V_t^+,\ \mathbb{P}-a.s.. \tag{4.37}$$

Noticing that the generator $\hat{F}$ is no longer Lipschitz in $|\hat{a}^{1/2}Z|$, in general, the statement above (4.37) is not straightforward if only the conditions that V is uniformly bounded on $(t,1]$ and that $V_r \to V_t^+$, $\mathbb{P}$-a.s. are given.

We define for each $r \in (t,1]$ a BSDE under $\mathbb{P}$ with parameters $(r, e^{\gamma V_r})$ and the generator $\hat{G}$ in the form of (4.20) and denote by $(\mathcal{Y}^{\mathbb{P}}(r,e^{\gamma V_r},\hat{G}), \mathcal{Z}^{\mathbb{P}}(r,e^{\gamma V_r},\hat{G}))$ its solution. From the relationship that

$$\mathcal{Y}_t^{\mathbb{P}}(r,e^{\gamma V_r},\hat{G}) = e^{\gamma y_t^{\mathbb{P}}(r,V_r)},\ 0 \leq t \leq r,\ \mathbb{P}-a.s.,$$

it suffices to prove the following statement instead of (4.37): there exists a sequence $\{r_m\}_{m\in\mathbb{N}} \subset (t,1]$ such that

$$\lim_{m\to+\infty} \mathcal{Y}_t^{\mathbb{P}}(r_m, e^{\gamma V_{r_m}}, \hat{G}) = e^{\gamma V_t^+},\ \mathbb{P}-a.s.. \tag{4.38}$$

Proceeding the same in the proof of Lemma 4.17, for each n, we consider the solution $\mathcal{Y}^{\mathbb{P}}(r,e^{\gamma V_r},\hat{G}^n)$ of the BSDE with parameters $(r, e^{\gamma V_r})$ and the generator $\hat{G}^n$ in the form of (4.22). Note $M := C(1 + \sup_{\omega\in\Omega} e^{\xi(\omega)})$ that is the uniform bound for all $\mathcal{Y}^{\mathbb{P}}(r,e^{\gamma V_r},\hat{G}^n)$, we have

$$E^{\mathbb{P}}[|\mathcal{Y}^{\mathbb{P}}(r,e^{\gamma V_r},\hat{G}^n) - e^{\gamma V_r}|^2] \leq C(1+n+\alpha_M)(r-t),$$

where n is the Lipschitz constant of $\hat{G}^n$ and $\alpha_M > 0$ depends only on M. Therefore, for a fixed $n \in \mathbb{N}$, there exists a sequence $\{r_m^n\}_{m\in\mathbb{N}} \subset (t,1]$ such that $r_m^n \downarrow t$ and

$$\lim_{m\to+\infty} |\mathcal{Y}_t^{\mathbb{P}}(r_m^n, e^{\gamma V_{r_m^n}}, \hat{G}^n) - e^{\gamma V_{r_m}}| = 0,\ \mathbb{P}-a.s.,$$

which implies

$$\begin{aligned}
&\lim_{m\to+\infty} |\mathcal{Y}_t^{\mathbb{P}}(r_m^n, e^{\gamma V_{r_m^n}}, \hat{G}^n) - e^{\gamma V_t^+}| \\
&\quad \leq \lim_{m\to+\infty} |\mathcal{Y}_t^{\mathbb{P}}(r_m^n, e^{\gamma V_{r_m^n}}, \hat{G}^n) - e^{\gamma V_{r_m^n}}| + \lim_{m\to+\infty} |e^{\gamma V_{r_m^n}} - e^{\gamma V_t^+}| = 0,\ \mathbb{P}-a.s..
\end{aligned}$$

By the diagonal argument, we could find a universal sequence $\{\tilde{r}_m\}_{m\in\mathbb{N}} \subset (t,1]$ such that $\tilde{r}_m \downarrow t$ and for each $n \in \mathbb{N}$,

$$\lim_{m\to+\infty} |\mathcal{Y}_t^{\mathbb{P}}(\tilde{r}_m, e^{\gamma V_{\tilde{r}_m}}, \hat{G}^n) - e^{\gamma V_t^+}| = 0,\ \mathbb{P}-a.s..$$

For each n, $m \in \mathbb{N}$,

$$|\mathcal{Y}_t^{\mathbb{P}}(\tilde{r}_m, e^{\gamma V_{\tilde{r}_m}}, \hat{G}^n)| \le M,$$

and Lemma 3.3 (monotone stability) in Morlais [64] shows that for each $m \in \mathbb{N}$, the following statement holds true $\mathbb{P}$-a.s.:

$$\mathcal{Y}_t^{\mathbb{P},n}(\tilde{r}_m, e^{\gamma V_{\tilde{r}_m}}, \hat{G}^n) \downarrow \mathcal{Y}_t^{\mathbb{P}}(\tilde{r}_m, e^{\gamma V_{\tilde{r}_m}}),\ \text{as } n \to +\infty.$$

Thus,

$$\begin{aligned}\lim_{m\to+\infty} \mathcal{Y}_t^{\mathbb{P}}(\tilde{r}_m, e^{\gamma V_{\tilde{r}_m}}) &= \lim_{m\to+\infty}\lim_{n\to+\infty} \mathcal{Y}_t^{\mathbb{P}}(\tilde{r}_m, e^{\gamma V_{\tilde{r}_m}}, \hat{G}^n)\\ &= \lim_{n\to+\infty}\lim_{m\to+\infty} \mathcal{Y}_t^{\mathbb{P}}(\tilde{r}_m, e^{\gamma V_{\tilde{r}_m}}, \hat{G}^n) = \lim_{n\to+\infty} e^{\gamma V_t^+} = e^{\gamma V_t^+},\end{aligned}$$

which ends the proof of (4.35).

Subsequently, the statement (4.36) could be proved in a similar way as Proposition 4.11 in Soner et al. [88]. □

Theorem 4.21 *Under (A1)-(A5) and for a given $\xi \in UC_b(\Omega)$, the 2BSDE (4.3) has a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$.*

Proof: From (4.33) and (4.36), we know that V is a càdlàg $\hat{F}$-supermartingale. Applying the Doob-Meyer type decomposition (cf. Theorem 5.8 in Ma and Yao [59]) under each $\mathbb{P} \in \mathcal{P}_H$,

$$V_t = V_1 + \int_t^1 \hat{F}_s(V_s, Z_s^{\mathbb{P}})ds - \int_t^1 Z_s^{\mathbb{P}} dB_s + K_1^{\mathbb{P}} - K_t^{\mathbb{P}},\ 0 \le t \le 1,\ \mathbb{P}-a.s.,$$

where $K^{\mathbb{P}}$ is a increasing process null at 0.

As shown in the proof of Theorem 4.5 in Soner et al. [88] one can find a universal Z such that for each $\mathbb{P} \in \mathcal{P}_H$,

$$Z_t = Z_t^{\mathbb{P}},\ 0 \le t \le 1,\ \mathbb{P}-a.s..$$

Defining $Y = V$, from (4.29), $Y \in \mathbb{D}_H^\infty$. Similarly to Lemma 3.1 in Possamaï and Zhou [78], we deduce that $Z \in \mathbb{H}_H^2$.

Then, it suffices to verify that the family of processes $\{K^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_H}$ satisfies the minimum condition (4.6). For a fixed $\mathbb{P} \in \mathcal{P}_H$, $t \in [0,1]$ and each $\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})$, using the notations from the proof of Theorem 4.13, we have

$$V_t - y_t^{\mathbb{P}'}(1,\xi) = E_t^{\mathbb{Q}'}\left[\int_t^1 M_s dK_s^{\mathbb{P}'}\right] \ge e^{-\mu} E_t^{\mathbb{Q}'}[K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}],$$

where

$$\left.\frac{d\mathbb{Q}'}{d\mathbb{P}'}\right|_{\mathcal{F}_t} = H_t^{\mathbb{P}'} := \int_0^t \kappa_s^{\mathbb{P}'} dW_s^{\mathbb{P}'},\ 0 \le t \le 1,\ \mathbb{P}'-a.s..$$

By Definition 4.5, $H := \{H^{\mathbb{P}'}\}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}$ is a $BMO(\mathcal{P}_H(t^+,\mathbb{P}))$-martingale. Applying Lemma 4.7 to the family $\mathcal{E}(H)$, there exists a $p > 1$ such that

$$\sup_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}\ \sup_{0\le t\le 1} \left\|E_t^{\mathbb{P}'}\left[\left(\frac{\mathcal{E}(H)_t}{\mathcal{E}(H)_1}\right)^{\frac{1}{p-1}}\right]\right\|_{L^\infty(\mathbb{P}')} \le C_E,$$

then

$$\begin{aligned} E_t^{\mathbb{P}'}[K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}] &\le E_t^{\mathbb{P}'}[\tfrac{\mathcal{E}(H^{\mathbb{P}'})_1}{\mathcal{E}(H^{\mathbb{P}'})_t}(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'})]^{\frac{1}{2p-1}} E_t^{\mathbb{P}'}[(\tfrac{\mathcal{E}(H^{\mathbb{P}'})_1}{\mathcal{E}(H^{\mathbb{P}'})_t})^{-\frac{1}{2p-2}}(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'})]^{\frac{2p-2}{2p-1}}\\ &\le E_t^{\mathbb{Q}'}[K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}]^{\frac{1}{2p-1}} E_t^{\mathbb{P}'}[(\tfrac{\mathcal{E}(H^{\mathbb{P}'})_1}{\mathcal{E}(H^{\mathbb{P}'})_t})^{-\frac{1}{p-1}}]^{\frac{p-1}{2p-1}} E_t^{\mathbb{P}'}[(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'})^2]^{\frac{p-1}{2p-1}}\\ &\le C_E^{\frac{p-1}{2p-1}} C_2^{\frac{p-1}{2p-1}} e^{\frac{\mu}{2p-1}} (V_t - y_t^{\mathbb{P}'}(1,\xi))^{\frac{1}{2p-1}}.\end{aligned}$$

From (4.35) and (4.36), we obtain

$$0 \leq \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}\in\mathcal{P}_H(t^+,\mathbb{P})} E_t^{\mathbb{P}'}[K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}] \leq C(V_t - \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}\in\mathcal{P}_H(t^+,\mathbb{P})} y_t^{\mathbb{P}'}(1,\xi))^{\frac{1}{2p-1}}$$
$$= C(V_t^+ - \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}\in\mathcal{P}_H(t^+,\mathbb{P})} y_t^{\mathbb{P}'}(1,\xi))^{\frac{1}{2p-1}} = 0,\ \mathbb{P}-a.s.,$$

which is the desired result. □

For each $\xi \in \mathcal{L}_H^\infty$, one can find a sequence $\{\xi^n\}_{n\in\mathbb{N}} \subset UC_b(\Omega)$, such that $||\xi^n - \xi||_{L_H^\infty} \to 0$. Thanks to a prior estimates, we have the following main result of the section.

Theorem 4.22 *Under (A1)-(A5) and for a given $\xi \in \mathcal{L}_H^\infty$, the 2BSDE (4.3) has a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$.*

Remark 4.23 *Similarly to the result in Matoussi et al. [61] for reflected 2BSDEs, applying Theorem 2.2 in Nutz [67], $\int ZdB$ could be defined universally if we add Zermelo-Fraenkel set theory with axiom of choice plus the continuum hypothesis into our framework. In this case, one can find a process K that is universal and $\mathbb{P}$-a.s. coincides with $K^{\mathbb{P}}$ under each $\mathbb{P} \in \mathcal{P}_H$, but is only $\mathcal{F}^*$-adapted, where*

$$\mathcal{F}_t^* := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}.$$

4.5 Application to finance

In this section, we re-solve some robust utility maximization problems introduced by Matoussi et al. [62].

4.5.1 Statement of the problem

The problem under consideration in Matoussi et al. [62] is to maximize in a robust way the expected utility of the terminal value of a portfolio on a financial market with some uncertainty on the objective probability and to choose an optimal trading strategy to attain this optimal goal under some restrictions.

This problem can be formulated as

$$V(x) := \sup_{\pi\in\tilde{\mathcal{A}}} \inf_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[U(X_T^\pi - \xi)], \tag{4.39}$$

where X_T^π is the terminal value of the wealth process associated with a strategy π from a given set $\tilde{\mathcal{A}}$ of all admissible trading strategies, ξ is a liability that matures at time T, U denotes the utility function and $\mathcal{P}$ is a set of all possible probability measures. Without loss of generality, we always assume that $T = 1$ in the sequel.

In this chapter, we study the problem that consists of non-dominated models, i.e., probability measures from the collection $\mathcal{P}$ could not be dominated by a finite measure. Consistent with the setting for 2BSDE theory, we assume that $\mathcal{P}$ is a subset of the class $\overline{\mathcal{P}}_S$ (cf. Definition 4.1), in which all the probability measures are mutually singular.

Definition 4.24 *In (4.39), let $\mathcal{P} = \tilde{\mathcal{P}}_H$ denote the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a} \leq \hat{a}_t \leq \overline{a} \text{ and } \hat{a}_t \in D_{F_t},\ \lambda \times \mathbb{P} - a.e.,$$

for some $\underline{a}$, $\overline{a} \in \mathbb{S}_d^{>0}$ and all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$.

Adapted to this setting of $\tilde{\mathcal{P}}_H$, we shall change a little our settings for quadratic 2BSDEs, that is, (A3) and (A5) will be replaced by the following (A3') and (A5'):

(A3') *F is continuous in (y, z) and has a quadratic growth, i.e., for each $(\omega, t, y, z, a) \in \Omega \times [0, 1] \times \mathbb{R} \times \mathbb{R}^d \times D_{F_t}$,*

$$|F_t(\omega, y, z, a)| \leq \alpha(a) + \beta(a)|y| + \frac{\gamma}{2}|a^{1/2}z|^2,$$

where γ is a strictly positive constant and α, β are non-negative deterministic functions satisfying that for some strictly positive constants $\overline{\alpha}$ and $\overline{\beta}$,

$$\alpha(\hat{a}_t) \leq \overline{\alpha}, \text{ and } \beta(\hat{a}_t) \leq \overline{\beta}, \;\; 0 \leq t \leq 1, \; \tilde{\mathcal{P}}_H - q.s..$$

(A5') *F is local Lipschitz in z, i.e., for each $(\omega, t, y, z, z', a) \in \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times D_{F_t}$,*

$$|F_t(\omega, y, z, a) - F_t(\omega, y, z', a)| \leq C(|a^{1/2}\phi(a)| + |a^{1/2}z| + |a^{1/2}z'|)|a^{1/2}(z - z')|,$$

where C is a strictly positive constant, ϕ is a deterministic function such that $\bar{\phi}(a) := a^{1/2}\phi(a)$ satisfies that for some strictly positive constant $\overline{\gamma}$,

$$|\bar{\phi}(\hat{a}_t)| = |\hat{a}_t^{1/2}\phi(\hat{a}_t)| \leq \overline{\gamma}, \; 0 \leq t \leq 1, \; \tilde{\mathcal{P}}_H - q.s..$$

Repeating all the proof for the wellposedness of quadratic 2BSDEs in the last section, we can have the following theorem:

Theorem 4.25 *Under (A1)- (A2), (A3'), (A4) and (A5') and for a given $\xi \in \mathcal{L}^{\infty}_{\hat{H}}$, the 2BSDE (4.3) has a unique solution $(Y, Z) \in \mathbb{D}^{\infty}_{H} \times \mathbb{H}^2_H$.*

We will have a detailed discussion for this kind of settings later in Subsection 4.5.4.

The financial market consists of one bond with zero interest rate and d stocks. The price processes of stocks is given by the following stochastic differential equation:

$$dS^i_t = S^i_t(b^i_t dt + dB^i_t), \; 0 \leq t \leq 1, \; i = 1, \ldots, d, \; \tilde{\mathcal{P}}_H - q.s.,$$

where B is a d-dimensional canonical process, b^i is an $\mathbb{R}$-valued process that is uniformly bounded by a constant $M > 0$ and is uniformly continuous in ω under the uniform norm $|| \cdot ||^{\infty}_1$, $i = 1, 2, \ldots, d$. By the definition of $\tilde{\mathcal{P}}_H$, for each $\mathbb{P} \in \tilde{\mathcal{P}}_H$,

$$B_t = \int_0^t \hat{a}_s^{1/2} dW^{\mathbb{P}}_s, \; 0 \leq t \leq 1, \; \mathbb{P} - a.s.,$$

where $\hat{a}^{1/2}$ plays in fact the role of volatility in (1) of Hu et al. [34]. Thus, the difference of $\hat{a}^{1/2}$ under each $\mathbb{P} \in \tilde{\mathcal{P}}_H$ allows us to model the volatility uncertainty.

In the following subsections, we study the problem (4.39) for two kinds of utility functions, the exponential and the power ones. For another type of utility function, i.e., the logarithmic one, Matoussi et al. has already solved the problem associated with it by a solution of a Lipschitz 2BSDE.

4.5.2 Robust exponential utility maximization

In this subsection, we consider the robust utility maximization problem (4.39) with an exponential utility function:

$$U(x) := -\exp(-cx), \; c > 0, \; x \in \mathbb{R}.$$

In this case, we denote $\pi = \{\pi_t\}_{0 \leq t \leq 1}$ the trading strategy, which is a d-dimensional $\mathcal{F}$-progressive measurable process. The ith component π^i_t describes the amount of money invested in stock i at time t, $i = 1, \ldots, d$, then, for a given trading strategy π, the wealth process X^π can be written as

$$X^\pi_t = x + \sum_{i=1}^{d} \int_0^t \frac{\pi^i_s}{S^i_s} dS^i_s = x + \int_0^t \pi_s(dB_s + b_s ds), \; 0 \leq t \leq 1, \; \tilde{\mathcal{P}}_H - q.s..$$

We now give the definition of admissible trading strategies.

Definition 4.26 *Let $\tilde{C}$ be a closed set in $\mathbb{R}^d$. The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all d-dimensional $\mathcal{F}$-progressively measurable processes $\pi = \{\pi_t\}_{0 \leq t \leq 1}$ that take values in $\tilde{C}$, $\lambda \otimes \mathcal{P}_H$-q.s., such that for each $\mathbb{P} \in \tilde{\mathcal{P}}_H$, $\int_0^1 |\hat{a}_t^{1/2}\pi_t|^2 dt < +\infty$, $\mathbb{P}$-a.s. and $\{\exp(-cX^\pi_\tau)\}_{\tau \in \mathcal{T}_0^1}$ is a $\mathbb{P}$-uniformly integrable family.*

Then, the utility maximization problem is equivalent to

$$V(x) := \sup_{\pi\in\tilde{\mathcal{A}}} \inf_{\mathbb{P}\in\tilde{\mathcal{P}}_H} E^{\mathbb{P}}\Big[-\exp\Big(-c\Big(x+\int_0^1 \pi_t(dB_t+b_tdt)-\xi\Big)\Big)\Big]. \tag{4.40}$$

We can also consider a reduced utility maximization problem under each $\mathbb{P}\in\tilde{\mathcal{P}}_H$, which is introduced by Theorem 7 in Hu et al. [34] and Theorem 4.1 in Morlais [64]. Following these well known results, one can find a $\pi^{\mathbb{P}^*}\in\tilde{\mathcal{A}}^{\mathbb{P}}$ that solves the reduced utility maximization problem:

$$V^{\mathbb{P}}(x) := \sup_{\pi\in\tilde{\mathcal{A}}^{\mathbb{P}}} E^{\mathbb{P}}\Big[-\exp\Big(-c\Big(x+\int_0^1 \pi_t(dB_t+b_tdt)-\xi\Big)\Big)\Big], \tag{4.41}$$

where $\tilde{\mathcal{A}}^{\mathbb{P}}$ is the collection of all admissible trading strategies given by Definition 1 in Hu et al. [34] under $\mathbb{P}$ and thus, $\tilde{\mathcal{A}}\subset\tilde{\mathcal{A}}^{\mathbb{P}}$. It is evident that

$$V(x)\le \inf_{\mathbb{P}\in\tilde{\mathcal{P}}_H} V^{\mathbb{P}}(x).$$

Therefore, the robust utility maximization problem (4.40) is solved if one can find an optimal strategy π^* such that

$$V(x)=\inf_{\mathbb{P}\in\tilde{\mathcal{P}}_H} E^{\mathbb{P}}\Big[-\exp\Big(-c\Big(x+\int_0^1 \pi^*_t(dB_t+b_tdt)-\xi\Big)\Big)\Big]=\inf_{\mathbb{P}\in\tilde{\mathcal{P}}_H} V^{\mathbb{P}}(x).$$

In what follows, we give the theorem similar to Theorem 4.1 in Matoussi et al. [62] but without some additional condition on ξ, b or on the border of $\tilde{C}$.

Theorem 4.27 *Assume that $\xi\in\mathcal{L}^\infty_{\tilde{H}}$. The value function of the utility maximization problem (4.40) is given by*

$$V(x)=-\exp(-c(x-Y_0)),$$

where Y_0 is defined by the unique solution $(Y,Z)\in\tilde{\mathbb{D}}^\infty_H\times\tilde{\mathbb{H}}^2_H$ of the following 2BSDE:

$$Y_t=\xi+\int_t^1 \hat{F}_s(Z_s)ds-\int_t^1 Z_sdB_s+K_1-K_t,\ 0\le t\le 1,\ \tilde{\mathcal{P}}_H-q.s., \tag{4.42}$$

where for each $(\omega,t,z,a)\in\Omega\times[0,1]\times\mathbb{R}^d\times\mathbb{S}^{>0}_d$,

$$F_t(\omega,z,a):=\frac{c}{2}\,\mathrm{dist}^2\left(a^{1/2}z+\frac{1}{c}a^{-1/2}b_t(\omega),a^{1/2}\tilde{C}\right)-z^{\mathbf{Tr}}b_t(\omega)-\frac{1}{2c}|a^{-1/2}b_t(\omega)|^2. \tag{4.43}$$

Moreover, there exists an optimal trading strategy $\pi^*\in\tilde{\mathcal{A}}$ with

$$\hat{a}_t^{1/2}\pi^*_t\in\Pi_{\hat{a}_t^{1/2}\tilde{C}}\Big(\hat{a}_t^{1/2}Z_t+\frac{1}{c}\hat{a}_t^{-1/2}b_t\Big),\ \lambda\otimes\tilde{\mathcal{P}}_H-q.s., \tag{4.44}$$

where $\Pi_A(r)$ denotes the collection of the elements in the closed set A that realize the minimal distance to the point r.

Remark 4.28 *Some of the assumptions adopted by Theorem 4.1 in Matoussi et al. [62] are removed: our assumptions for the wellposedness of quadratic 2BSDEs does not concern the size of ξ, so we do not need to assume in addition that the liability ξ is small enough in norm; on the other hand, we do not have any requirement on the regularity of the derivatives of the generator $\hat{F}$ and thus, the border of $\tilde{C}$ is no longer assumed to be a $\mathcal{C}^2$ curve. It is evident that these two additional assumptions have limitations in real financial market: the one on ξ is not practical; the other one on the border of $\tilde{C}$ is often difficult to verify.*

Sketch of the proof of Theorem 4.27: We prove this theorem by following procedures adopted by Matoussi et al. [62] but with some modifications, and we only give the sketch.

Step 1: In this step, we show that the 2BSDE (4.42) has a unique solution by verifying that the generator F satisfies (A1)-(A2), (A3') and (A5'). Then, Theorem 4.25 states that the 2BSDE (4.42) admits a unique solution $(Y,Z)\in\tilde{\mathbb{D}}^\infty_H\times\tilde{\mathbb{H}}^2_H$.

- From the assumptions that b is uniform bounded and that $\tilde{C}$ is closed, we have, for each $(\omega, z) \in \Omega \times \mathbb{R}^d$, $D_{F_t(\omega,z)} = \mathbb{S}_d^{>0}$, which implies that (A1) is satisfied.
- Since b is $\mathcal{F}$-progressive measurable and uniformly continuous in ω under the uniform norm, for each $(z, a) \in \mathbb{R}^d \times \mathbb{S}_d^{>0}$, $F(z, a)$ is $\mathcal{F}$-progressive measurable and uniformly continuous in ω.
- For each $a \in \mathbb{S}_d^+$ that satisfies $\underline{a} \le a \le \overline{a}$, there exist a $\overline{K} > 0$ that depends only on $\overline{a}$ and $\tilde{C}$ such that

$$\inf\{|r| : r \in a^{1/2}\tilde{C}\} \le \overline{K},$$

and another $\underline{K} > 0$ that depends only on $\underline{a}$ and M, such that for each $\omega \in \Omega$,

$$|a^{-1/2}b_t(\omega)|^2 \le \mathrm{tr}(a^{-1})M^2 = \underline{K}^2.$$

Then, for each $(t, z) \in [0, 1] \times \mathbb{R}^d$,

$$\mathrm{dist}^2\left(a^{1/2}z + \frac{1}{c}a^{-1/2}b_t, a^{1/2}\tilde{C}\right) \le 2|a^{1/2}z|^2 + 2\left(\frac{1}{c}|a^{-1/2}b_t| + \overline{K}\right)^2, \tag{4.45}$$

from which we deduce

$$|F_t(\omega, z, a)| \le \left(2c\overline{K}^2 + \frac{5+c}{2c}\underline{K}^2\right) + \left(\frac{1}{2} + c\right)|a^{1/2}z|^2.$$

That is to say (A3') is satisfied.
- For each $(t, z^1, z^2) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ and $a \in \mathbb{S}_d^{>0}$ that satisfies $\underline{a} \le a \le \overline{a}$,

$$\begin{aligned} F_t(\omega, z^1, a) - F_t(\omega, z^2, a) = \frac{c}{2}\Bigg(& \mathrm{dist}^2\left(a^{1/2}z^1 + \frac{1}{c}a^{-1/2}b_t, a^{1/2}\tilde{C}\right) \\ & - \mathrm{dist}^2\left(a^{1/2}z^2 + \frac{1}{c}a^{-1/2}b_t, a^{1/2}\tilde{C}\right)\Bigg) - (z^1 - z^2)^{\mathbf{Tr}}b_t. \end{aligned}$$

By the Lipschitz property of the distance function with respect to a closed set, we obtain the following inequality:

$$|F_t(\omega, z^1, a) - F_t(\omega, z^2, a)| \le \frac{c}{2}\left(\left(2\overline{K} + \frac{4}{c}\underline{K}\right) + |a^{1/2}z^1| + |a^{1/2}z^2|\right)|a^{1/2}(z^1 - z^2)|,$$

from which (A5') is satisfied.

Step 2: We define, for each $\pi \in \tilde{\mathcal{A}}$,

$$R_t^\pi = -\exp(-c(X_t^\pi - Y_t)),\ 0 \le t \le 1, \tag{4.46}$$

where Y is the solution to 2BSDE (4.42). Then, we decompose R^π into a product of two processes, i.e., $R^\pi = M^\pi A^\pi$, where for each $\mathbb{P} \in \tilde{\mathcal{P}}_H$,

$$\begin{aligned} M_t^\pi := e^{-c(x-Y_0)}\exp\Bigg(& -\int_0^t c(\pi_s - Z_s)dB_s \\ & - \frac{1}{2}\int_0^t c^2|\hat{a}_s^{1/2}(\pi_s - Z_s)|^2 ds - cK_t^{\mathbb{P}}\Bigg),\ 0 \le t \le 1,\ \mathbb{P} - a.s., \end{aligned}$$

and

$$A_t^\pi := -\exp\left(-\int_0^t \left(c\pi_s^{\mathbf{Tr}}b_s + c\hat{F}_s(Z_s) - \frac{1}{2}c^2|\hat{a}_s^{1/2}(\pi_s - Z_s)|^2\right)ds\right),\ 0 \le t \le 1,\ \mathbb{P} - a.s..$$

We rewrite A^π into the following form,

$$\begin{aligned} A_t^\pi = -\exp\Bigg(-\int_0^t \Bigg(& \frac{c^2}{2}\left|\hat{a}_s^{1/2}\pi_s - \left(\hat{a}_s^{1/2}Z_s + \frac{1}{c}\hat{a}_s^{-1/2}b_s\right)\right|^2 \\ & - cZ_s^{\mathbf{Tr}}b_s - \frac{1}{2}|\hat{a}_s^{-1/2}b_s|^2 - c\hat{F}_s(Z_s)\Bigg)ds\Bigg). \end{aligned}$$

It is readily to observe that if $\pi = \pi^*$ that satisfies (4.44), then

$$A_t^{\pi^*} \equiv -1,\ 0 \le t \le 1,\ \mathbb{P} - a.s..$$

Moreover, Lemma 11 in Hu et al. [34] says that one can define such a π^* that is $\mathcal{F}$-progressively measurable if Z is $\mathcal{F}$-progressively measurable.

In the previous section, we have already proved that $Z \in \tilde{\mathbb{H}}^2_{BMO(\tilde{\mathcal{P}}_H)}$. To show that $\pi^* - Z \in \tilde{\mathbb{H}}^2_{BMO(\tilde{\mathcal{P}}_H)}$, it suffices to verify that π^* is also in $\tilde{\mathbb{H}}^2_{BMO(\tilde{\mathcal{P}}_H)}$. Applying triangle inequality to $|\hat{a}_t^{1/2}\pi_t^*|$ and recalling (4.45), we have, for each $t \in [0,1]$,

$$\begin{aligned}
|\hat{a}_t^{1/2}\pi_t^*| &\le \left|\hat{a}_t^{1/2}Z_t + \frac{1}{c}\hat{a}_t^{-1/2}b_t\right| + \left|\hat{a}_t^{1/2}\pi_t^* - \left(\hat{a}_t^{1/2}Z_t + \frac{1}{c}\hat{a}_t^{-1/2}b_t\right)\right| \\
&\le |\hat{a}_t^{1/2}Z_t| + \frac{1}{c}|\hat{a}_t^{-1/2}b_t| + \text{dist}\left(a_t^{1/2}Z_t + \frac{1}{c}\hat{a}_t^{-1/2}b_t, \hat{a}^{1/2}\tilde{C}\right) \\
&\le 2|\hat{a}_t^{1/2}Z_t| + \frac{2}{c}\underline{K} + 2\overline{K},\ \mathcal{P}_H - q.s.,
\end{aligned} \tag{4.47}$$

which implies that π^* is an element in $\tilde{\mathbb{H}}^2_{BMO(\tilde{\mathcal{P}}_H)}$.

As $\pi^* \in \tilde{\mathbb{H}}^2_{BMO(\tilde{\mathcal{P}}_H)}$, for each $\mathbb{P} \in \tilde{\mathcal{P}}_H$, $\hat{a}^{1/2}\pi^*$ is a $BMO(\mathbb{P})$-martingale generator. By Remark 8 in Hu et al. [34], $\{\exp -cX_\tau^\pi\}_{\tau\in\mathcal{T}_0^1}$ is a $\mathbb{P}$-uniformly integrable family and it is easy to verify that $E^{\mathbb{P}}[\int_0^1 |\hat{a}_t^{1/2}\pi_t^*|^2 dt] < +\infty$. Thus, $\pi^* \in \tilde{\mathcal{A}}$.

Step 3: We now prove that for each $\mathbb{P} \in \tilde{\mathcal{P}}_H$,

$$\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{P}'}[M_1^{\pi^*}] = M_t^{\pi^*},\ 0 \le t \le 1,\ \mathbb{P} - a.s., \tag{4.48}$$

so that

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{P}'}[R_1^{\pi^*}] = R_t^{\pi^*},\ 0 \le t \le 1,\ \mathbb{P} - a.s.. \tag{4.49}$$

Since $-c(\pi^* - Z)$ is a $BMO(\tilde{\mathcal{P}}_H)$-martingale generator, under each $\mathbb{P}' \in \tilde{\mathcal{P}}_H(t,\mathbb{P})$, $\mathcal{E}(-c\int_0^\cdot(\pi_t^* - Z_t)dB_t)$ is an exponential martingale, and M^{π^*} can be regarded as a product of a martingale and a positive decreasing process. Thus, it is easy to show that for each $\mathbb{P} \in \tilde{\mathcal{P}}_H$,

$$\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{P}'}[M_1^{\pi^*}] \le M_t^{\pi^*},\ 0 \le t \le 1,\ \mathbb{P} - a.s.. \tag{4.50}$$

To get the desired result, it suffices to prove the reverse inequality. Noticing that $M_1^{\pi^*}$ and $M_t^{\pi^*}$ are both positive, we can consider the ratio $\frac{M_1^{\pi^*}}{M_t^{\pi^*}}$. We calculate for each $t \in [0,1]$ and $\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})$,

$$\begin{aligned}
\frac{M_1^{\pi^*}}{M_t^{\pi^*}} = \exp\Bigg(&- \int_t^1 c(\pi_s^* - Z_s)dB_s \\
&- \frac{1}{2}\int_t^1 c^2|\hat{a}_s^{1/2}(\pi_s^* - Z_s)|^2 ds - c(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'})\Bigg),\ \mathbb{P}' - a.s..
\end{aligned}$$

Changing measure by

$$\left.\frac{d\mathbb{Q}'}{d\mathbb{P}'}\right|_{\mathcal{F}_t} = \mathcal{E}\bigg(- c\int_0^\cdot (\pi_s^* - Z_s)\hat{a}_s^{1/2}dW_s^{\mathbb{P}'}\bigg)_t,$$

we have

$$E_t^{\mathbb{P}'}\left[\frac{M_1^{\pi^*}}{M_t^{\pi^*}}\right] = E_t^{\mathbb{Q}'}[\exp(-c(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}))],\ \mathbb{P}' - a.s..$$

By Jensen's inequality and the convexity of $\exp(-cx)$, we obtain

$$\begin{aligned}\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{P}'}\left[\frac{M_1^{\pi^*}}{M_t^{\pi^*}}\right] &= \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{Q}'}[\exp(-c(K_1^{\mathbb{P}'}-K_t^{\mathbb{P}'}))]\\ &\geq \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} \exp(-cE_t^{\mathbb{Q}'}[K_1^{\mathbb{P}'}-K_t^{\mathbb{P}'}])\\ &\geq \exp(-c \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{Q}'}[K_1^{\mathbb{P}'}-K_t^{\mathbb{P}'}]).\end{aligned}$$

Similar to (4.15), we know, for some $p, q > 1$ that satisfy $1/p + 1/q = 1$,

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{Q}'}[K_1^{\mathbb{P}'}-K_t^{\mathbb{P}'}] \leq C_{RH}^{1/q}C_{2p-1}^{1/2p} \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{P}'}[K_1^{\mathbb{P}'}-K_t^{\mathbb{P}'}]^{1/2p} = 0,$$

where C_{RH} is the constant in Lemma 4.6 and C_{2p-1} is from (4.10). The inequality above implies that

$$\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{P}'}\left[\frac{M_1^{\pi^*}}{M_t^{\pi^*}}\right] \geq 1. \tag{4.51}$$

Then, (4.48) comes after (4.50) and (4.51).

Step 4: Under each $\mathbb{P} \in \tilde{\mathcal{P}}_H$, the canonical process B is a $\mathbb{P}$-martingale and $\hat{F}_t(z)$ is in fact (2.6) in Morlais [64]. Thus, the value function of the reduced utility maximization problem is given by

$$V^{\mathbb{P}}(x) = -\exp(-c(x - Y_0^{\mathbb{P}})),$$

where $Y_0^{\mathbb{P}}$ is defined by the unique solution $(Y^{\mathbb{P}}, Z^{\mathbb{P}}) \in D^\infty(\mathbb{P}) \times H^2(\mathbb{P})$ of the following BSDE:

$$Y_t^{\mathbb{P}} = \xi + \int_t^1 \hat{F}_s(Z_s)ds - \int_t^1 Z_s dB_s,\ 0 \leq t \leq 1,\ \mathbb{P}-a.s.. \tag{4.52}$$

By Theorem 3.2, we have

$$Y_0 = \sup_{\mathbb{P}\in\tilde{\mathcal{P}}_H} Y_0^{\mathbb{P}}.$$

From (4.49) and (4.52), it holds true that

$$\begin{aligned}\inf_{\mathbb{P}\in\tilde{\mathcal{P}}_H} E^{\mathbb{P}}[-\exp(-c(X_t^{\pi*}-\xi))] &= \inf_{\mathbb{P}\in\tilde{\mathcal{P}}_H} E^{\mathbb{P}}[R_1^{\pi^*}] = R_0^{\pi^*}\\ &= -\exp(-c(x-Y_0)) = \inf_{\mathbb{P}\in\tilde{\mathcal{P}}_H} -\exp(-c(x-Y_0^{\mathbb{P}})),\end{aligned}$$

which implies that π^* is the optimal strategy. We complete the proof. □

Remark 4.29 *In fact, we adopt a weaker assumption on the admissible strategy than the one in Theorem 4.1 in Matoussi et al. [62]. We only assume that π is an admissible strategy defined by Hu et al. [34] and Morlais [64] under each $\mathbb{P} \in \tilde{\mathcal{P}}_H$, i.e.,*

$$\tilde{\mathcal{A}} = \bigcap_{\mathbb{P}\in\tilde{\mathcal{P}}_H} \tilde{\mathcal{A}}^{\mathbb{P}},$$

while Matoussi et al. [62] assumed that $\pi \in \tilde{\mathbb{H}}^2_{BMO(\tilde{\mathcal{P}}_H)}$. Under this stronger assumption, all R^π satisfies the minimal condition (4.49) and they verified that $A^\pi \leq A^* \equiv -1$, for all π is admissible, so that π^* is optimal. In this chapter, we justify that π^* is optimal for this larger set of admissible strategies by a min-max property as we showed in Step 4, which is regardless of whether the admissible strategy other than the optimal one is a $BMO(\tilde{\mathcal{P}}_H)$-martingale generator. Although $\pi^* \in \tilde{\mathbb{H}}^2_{BMO(\tilde{\mathcal{P}}_H)}$, this result is more general.

4.5.3 Robust power utility maximization

In this subsection, we consider the problem (4.39) with a power utility function:

$$U(x) := \frac{1}{\gamma}x^\gamma,\ \gamma < 1,\ x \in \mathbb{R}.$$

In this case, a d-dimensional $\mathcal{F}$-progressively measurable process $\{\rho_t\}_{0\le t\le 1}$ denotes the trading strategy, whose component ρ_t^i describes the proportion of money invested in stock i at time t, $i = 1, 2, \dots, d$, then, for a given trading strategy ρ, the wealth process X^ρ can be written as

$$X_t^\rho = x + \sum_{i=1}^d \int_0^t \frac{X_s^\rho \rho_s^i}{S_s^i} dS_s^i = x + \int_0^t X_s^\rho \rho_s (dB_s + b_s ds),\ 0 \le t \le 1,\ \tilde{\mathcal{P}}_H - q.s.,$$

where the initial capital x is positive. Then, X^ρ is given by

$$X_t^\rho := x\mathcal{E}\bigg(\int_0^\cdot \rho_s(dB_s + b_s ds)\bigg)_t,\ 0 \le t \le 1,$$

under each $\mathbb{P} \in \tilde{\mathcal{P}}_H$.

Definition 4.30 *Let $\tilde{C}$ be a closed set in $\mathbb{R}^d$. The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all d-dimensional $\mathcal{F}$-progressively measurable processes $\rho = \{\rho_t\}_{0\le t\le 1}$ that take values in $\tilde{C}$, $\lambda \otimes \tilde{\mathcal{P}}_H$-q.s. and for each $\mathbb{P} \in \tilde{\mathcal{P}}_H$, $\int_0^1 |\hat{a}_t^{1/2}\rho_t|^2 dt < +\infty$, $\mathbb{P}$-a.s..*

For each $\mathbb{P} \in \tilde{\mathcal{P}}_H$, we define a probability measure $\mathbb{Q} \ll \mathbb{P}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \mathcal{E}\bigg(-\int_0^\cdot b_s^{\mathbf{Tr}}\hat{a}_s^{1/2} dW_s^{\mathbb{P}}\bigg)_t,$$

then, by the definition above, for each $\rho \in \tilde{\mathcal{A}}$, X^ρ is a $\mathbb{Q}$-local martingale bounded from below. Thus, X^ρ is a $\mathbb{Q}$-supermartingale. Since $\mathbb{Q} \ll \mathbb{P}$, the strategy ρ is free of arbitrage under $\mathbb{P}$.

We suppose that the investor has no liability, i.e., $\xi = 0$, then the maximization problem is equivalent to

$$V(x) := \frac{1}{\gamma}x^\gamma \sup_{\rho\in\tilde{\mathcal{A}}} \inf_{\mathbb{P}\in\tilde{\mathcal{P}}_H} E^{\mathbb{P}}\bigg[\exp\bigg(\gamma\int_0^1 \rho_s(dB_s + b_s ds) - \frac{\gamma}{2}\int_0^1 |\hat{a}_s^{1/2}\rho_s|^2 ds\bigg)\bigg]. \tag{4.53}$$

Similarly to that in the last subsection, we have the following theorem:

Theorem 4.31 *The value function of the utility maximization problem (4.53) is given by*

$$V(x) = \frac{1}{\gamma}x^\gamma \exp(Y_0),$$

where Y_0 is defined by the unique solution $(Y, Z) \in \tilde{\mathbb{D}}_H^\infty \times \tilde{\mathbb{H}}_H^2$ of the following 2BSDE:

$$Y_t = 0 + \int_t^1 \hat{F}_s(Z_s)ds - \int_t^1 Z_s dB_s + K_1 - K_t,\ 0 \le t \le 1,\ \tilde{\mathcal{P}}_H - q.s., \tag{4.54}$$

where for each $(\omega, t, z, a) \in \Omega \times [0,1] \times \mathbb{R}^d \times \mathbb{S}_d^{>0}$,

$$F_t(\omega, z, a) := -\frac{\gamma(1-\gamma)}{2}\operatorname{dist}^2\bigg(\frac{1}{1-\gamma}(a^{1/2}z + a^{-1/2}b_t(\omega)), a_t^{1/2}\tilde{C}\bigg) + \frac{\gamma|a^{1/2}z + a^{-1/2}b_t(\omega)|^2}{2(1-\gamma)} + \frac{1}{2}|a^{1/2}z|^2. \tag{4.55}$$

Moreover, there exists an optimal trading strategy $\rho^* \in \tilde{\mathcal{A}}$ with

$$\hat{a}_t^{1/2}\rho_t^*(\omega) \in \Pi_{\hat{a}_t^{1/2}\tilde{C}}\bigg(\frac{1}{1-\gamma}(\hat{a}_t^{1/2}z + \hat{a}_t^{-1/2}b_t(\omega))\bigg),\ 0 \le t \le 1,\ \tilde{\mathcal{P}}_H - q.s., \tag{4.56}$$

where $\Pi_A(r)$ denotes the collection of the elements in the closed set A that realize the minimal distance to the point r.

Sketch of the proof: Following similar procedures in the proof of Theorem 4.27, we verify that the generator F in 2BSDE (4.54) satisfies (A1)-(A2), (A3') and (A5') and define a family of processes $\{R^\rho\}_{\rho\in\tilde{\mathcal{A}}}$ by

$$R_t^\rho := \frac{1}{\gamma}x^\gamma \exp\left(\gamma\int_0^t \rho_s(dB_s + b_s ds) - \frac{\gamma}{2}\int_0^t |\hat{a}_s^{1/2}\rho_s|^2 ds + Y_t\right),\ 0\le t\le 1,$$

such that for each $\mathbb{P}\in\tilde{\mathcal{P}}_H$,

- R_0^ρ is a constant independent of ρ;
- $R_1^\rho = \frac{1}{\gamma}(X_1^\rho)^\gamma$, for each $\rho\in\tilde{\mathcal{A}}$.

Then, we rewrite R^ρ under each $\mathbb{P}\in\tilde{\mathcal{P}}_H$ as the following:

$$R_t^\rho = \frac{1}{\gamma}x^\gamma \exp(Y_0)\mathcal{E}\left(\int_0^\cdot (\gamma\rho_s + Z_s)dB_s\right)_t e^{-K_t^{\mathbb{P}}}\exp\left(\int_0^t \nu_s ds\right),\ 0\le t\le 1,\ \mathbb{P}-a.s.,$$

where

$$\nu_t := -\frac{\gamma(1-\gamma)}{2}\left|\hat{a}_t^{1/2}\rho_t - \frac{1}{1-\gamma}(\hat{a}_t^{1/2}Z_t + \hat{a}_t^{-1/2}b_t)\right|^2 + \frac{\gamma|\hat{a}_t^{1/2}Z_t + \hat{a}_t^{-1/2}b_t|^2}{2(1-\gamma)} + \frac{1}{2}|\hat{a}_t^{1/2}Z_t|^2 - \hat{F}_t(Z_t).$$

Similarly to (4.49), we could find an optimal strategy ρ^* such that for each $\mathbb{P}\in\tilde{\mathcal{P}}_H$,

$$\nu_t^{\rho^*} \equiv 0,\ 0\le t\le 1,\ \mathbb{P}-a.s.$$

and thus,

$$\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\tilde{\mathcal{P}}_H(t,\mathbb{P})} E_t^{\mathbb{P}'}[R_1^{\rho^*}] = R_t^{\rho^*},\ 0\le t\le 1,\ \mathbb{P}-a.s.. \tag{4.57}$$

The desired result comes after (4.57) and the min-max property. □

Remark 4.32 *In Matoussi et al. [62], only the case that $\gamma < 0$ was considered. According to their assumption that $\tilde{C}$ contains 0, we calculate*

$$\hat{F}_t^0 = -\frac{\gamma}{2(1-\gamma)}|a_t^{-1/2}b_t|^2,$$

where $-\frac{\gamma}{2(1-\gamma)}$ is dominated by $\frac{1}{2}$ when $\gamma<0$ and so that the assumption that b is small can imply that F^0 is small enough.

4.5.4 Some remarks on the class of probability measures and assumptions

We have already seen that 2BSDEs (4.42) and (4.54) are discussed under some new settings, where $\mathcal{P}_H$ was changed into $\tilde{\mathcal{P}}_H$; (A3) and (A5) were changed into (A3') and (A5'). In what follows, we would like to discuss more about these conditions and the class of probability measures.

Since these weakened conditions shall be related to some given series of probability measure classes $\{\mathcal{P}_H^t\}_{t\in[0,1]}$, we first give the following definition:

Definition 4.33 *We say a series of probability measure classes $\{\mathcal{P}_H^t\}_{t\in[0,1]}$ is consistent if the following points are satisfied (we note $\mathcal{P}_H^0 = \mathcal{P}_H$.):*

- *For each $\mathbb{P}\in\mathcal{P}_H$, for $\mathbb{P}$-a.s. $\omega\in\Omega$ and each $\tau\in\mathcal{T}_0^1$, $\mathbb{P}^{\tau,\omega}\in\mathcal{P}_H^{\tau(\omega)}$;*
- *For each $\tau\in\mathcal{T}_0^1$, $A\in\mathcal{F}_\tau$, $\mathbb{P}\in\mathcal{P}_H$ and $\hat{\mathbb{P}}^\tau\in\mathcal{P}_H^\tau$, $\mathbb{P}\otimes_\tau^A\hat{\mathbb{P}}^\tau\in\mathcal{P}_H$, where for each $E\subset\Omega$,*

$$\mathbb{P}\otimes_\tau^A\hat{\mathbb{P}}^\tau(E) := E^{\mathbb{P}}[E^{\hat{\mathbb{P}}^\tau}[(\mathbf{1}_E)^{\tau,\omega}]\mathbf{1}_A] + \mathbb{P}(E\cap A^c).$$

In the 2BSDE framework, the series of classes defined by Definition 4.1 is consistent, since the first point is guaranteed by Lemma 4.1 in Soner et al. [88] and the second one is in fact the reduced version ($n = 1$) of the statement (4.19) in Soner et al. [88]. These two properties play an important role in our proof of the dynamic programming principle (cf. Proposition 4.19).

In what follows, we verify that the series of classes defined by Definition 4.24 is consistent. In this case, $\tilde{\mathcal{P}}_H^t$ consists of all those $\mathbb{P} \in \overline{\mathcal{P}}_S^t$ such that

$$\underline{a} \le \hat{a}_s^t \le \overline{a} \text{ and } \hat{a}_s^t \in D_{F_s},\ \lambda \times \mathbb{P}^t - a.e.,$$

for some $\underline{a}, \overline{a} \in \mathbb{S}_d^{>0}$ and each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. Since $\tilde{\mathcal{P}}_H \subset \overline{\mathcal{P}}_S$, by Lemma 4.1 in Soner et al. [88], for a given $\mathbb{P} \in \tilde{\mathcal{P}}_H$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathbb{P}^{\tau,\omega} \in \overline{\mathcal{P}}_S^{\tau(\omega)}$ and

$$\underline{a} \le \hat{a}_t^{\tau(\omega)}(\tilde{\omega}) = \hat{a}_t^{\tau,\omega}(\tilde{\omega}) = \hat{a}_t(\omega \otimes^\tau \tilde{\omega}) \le \overline{a},\ \lambda \times \mathbb{P}^{\tau,\omega} - a.e..$$

On the other hand, the proof of statement (4.19) in Soner et al. [88] showed that $\mathbb{P} \otimes_\tau^A \hat{\mathbb{P}}^\tau \in \overline{\mathcal{P}}_S$. Defining $\tilde{\mathbb{P}} := \mathbb{P} \otimes_\tau^A \hat{\mathbb{P}}^\tau$, it suffices to verify that

$$\underline{a} \le \hat{a}_t \le \overline{a},\ \lambda \times \tilde{\mathbb{P}} - a.e.. \tag{4.58}$$

We calculate

$$\int_0^1 E^{\tilde{\mathbb{P}}}[\mathbf{1}_{\{\hat{a}_t \notin [\underline{a},\overline{a}]\}}]dt = \int_0^1 (E^{\mathbb{P}}[E^{\hat{\mathbb{P}}^\tau}[(\mathbf{1}_{\{\hat{a}_t \notin [\underline{a},\overline{a}]\}})^{\tau,\omega}]\mathbf{1}_A] + E^{\mathbb{P}}[\mathbf{1}_{\{\{\hat{a}_t \notin [\underline{a},\overline{a}]\} \cap A^c\}}])dt,$$

where

$$E^{\hat{\mathbb{P}}^\tau}[(\mathbf{1}_{\{\hat{a}_t \notin [\underline{a},\overline{a}]\}})^{\tau,\omega}]\mathbf{1}_A(\omega) = \begin{cases} E^{\hat{\mathbb{P}}^\tau}[\mathbf{1}_{\{\hat{a}_t^\tau \notin [\underline{a},\overline{a}]\}}(\tilde{\omega})] = 0 & , \quad \omega \in A,\ t \ge \tau(\omega); \\ \mathbf{1}_{\{\hat{a}_t \notin [\underline{a},\overline{a}]\}}(\omega) & , \quad \omega \in A,\ t < \tau(\omega); \\ 0 & , \quad otherwise. \end{cases}$$

Thus,

$$\int_0^1 E^{\tilde{\mathbb{P}}}[\mathbf{1}_{\{\hat{a}_t \notin [\underline{a},\overline{a}]\}}]dt \le \int_0^1 E^{\mathbb{P}}[\mathbf{1}_{\{\hat{a}_t \notin [\underline{a},\overline{a}]\}}]dt = 0,$$

which implies (4.58).

Remark 4.34 *Suppose that a consistent series of probability measure classes $\{\mathcal{P}_H^t\}_{t\in[0,1]} \subset \overline{\mathcal{P}}_S$ is given (not limited to the form defined by Definition 4.1 and 4.24), then (A3) can be even weakened to the following form, which is similar to (H1) in Morlais [64] for quadratic BSDEs:*

(A3") *F is continuous in (y, z) and has a quadratic growth in z, i.e., for each $(\omega, t, y, z, a) \in \Omega \times [0, 1] \times \mathbb{R} \times \mathbb{R}^d \times D_{F_t}$,*

$$|F_t(\omega, y, z, a)| \le \alpha_t(a) + \beta_t(a)|y| + \frac{\gamma}{2}|a^{1/2}z|^2, \tag{4.59}$$

where γ is a strictly positive constant and α, β satisfy that

- *For each $a \in \mathbb{S}_d^{>0}$, $\alpha(a)$ and $\beta(a)$ are positive $\mathcal{F}$-progressive measurable processes;*
- *For some $\overline{\alpha}$ and $\overline{\beta}$, which are strictly positive constants,*

$$\int_0^1 \alpha_t(\hat{a}_t)dt \le \overline{\alpha} \text{ and } \int_0^1 \beta_t(\hat{a}_t)dt \le \overline{\beta},\ \mathcal{P}_H - q.s.;$$

- *For each $(\omega, t) \in \Omega \times (0, 1]$ and $\mathbb{P}^t \in \mathcal{P}_H^t$,*

$$\int_t^1 \alpha_s^{t,\omega}(\hat{a}_s^t)ds \le \overline{\alpha} \text{ and } \int_t^1 \beta_s^{t,\omega}(\hat{a}_s^t)ds \le \overline{\beta},\ \mathbb{P}^t - a.s.,$$

  where $\overline{\alpha}$, $\overline{\beta}$ are the same as above.

We recall (4.28) that for each $(\omega, t) \in \Omega \times [0,1]$, $V_t(\omega)$ concerns solutions of (t,ω)-shifted quadratic BSDEs under all $\mathbb{P}^t \in \mathcal{P}_H^t$. Therefore, for each $t \in [0,1]$, at least $\mathcal{P}_H$-q.s. $\omega \in \Omega$, (t,ω)-shifted generator should satisfy (H1) in Morlais [64] (or similar conditions for quadratic BSDEs) under each $\mathbb{P}^t \in \mathcal{P}_H^t$ to ensure the existence of these solutions. We notice that the original condition (A3) is posed pathwisely, that is, it holds for all $(\omega, t) \in \Omega \times [0,1]$, whereas (A3") also involves pathwise settings for each $(\omega, t) \in \Omega \times [0,1]$. Therefore, (4.28) can be well defined under these two conditions.

A natural question arises: if (4.4) and (4.59) can be written in a $\mathcal{P}_H$-q.s. version; if the third point of (A3") can be removed?

We consider the first question: suppose that for all $(t,y,z,a) \in [0,1] \times \mathbb{R} \times \mathbb{R}^d$,

$$|\hat{F}_t(y,z)| \le \alpha + \beta|y| + \frac{\gamma}{2}|\hat{a}_t^{1/2} z|^2,\ \mathcal{P}_H - q.s..$$

Fixing an $\mathbb{P}^t \in \mathcal{P}_H^t$, we can choose an arbitrage $\mathbb{P} \in \mathcal{P}_H$ and construct a concatenation probability $\hat{\mathbb{P}} := \mathbb{P} \otimes_t^\Omega \mathbb{P}^t$. Since $\{\mathcal{P}_H^t\}_{t\in[0,1]}$ is consistent, $\hat{\mathbb{P}} \in \mathcal{P}_H$, $\mathbb{P}|_{\mathcal{F}_t} = \hat{\mathbb{P}}|_{\mathcal{F}_t}$ and for each $\omega \in \Omega$, $\hat{\mathbb{P}}^{t,\omega} = \mathbb{P}^t$. Thus, we have for $\mathbb{P}$-a.s, $\omega \in \Omega$ and all $(s,y,z,a) \in [t,1] \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} |\hat{F}_s^{t,\omega}(y,z)| &= |F_s(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s(\omega \otimes_t \tilde{\omega})) \\ &\le \alpha + \beta|y| + \frac{\gamma}{2}|\hat{a}_s^{1/2}(\omega \otimes_t \tilde{\omega}) z|^2 = \alpha + \beta|y| + \frac{\gamma}{2}|(\hat{a}_s^t)^{1/2}(\tilde{\omega}) z|^2,\ \mathbb{P}^t - a.s.. \end{aligned} \tag{4.60}$$

Since $\mathbb{P}$ is arbitrage, we can deduce that for $\mathcal{P}_H$-q.s. $\omega \in \Omega$, (4.60) is satisfied. In other words, defining for each $\mathbb{P}^t \in \mathcal{P}_H^t$ a set:

$$E^{\mathbb{P}^t} := \{\omega :\ \hat{F}_s^{t,\omega}(y,z) \text{ satisfies (4.60)},\ \mathbb{P}^t - a.s.\},$$

we have $\mathbb{P}(E^{\mathbb{P}^t}) = 1$ for all $\mathbb{P} \in \mathcal{P}_H$. At the end of the day, we still have no idea about $\mathbb{P}(\cap_{\mathbb{P}^t \in \mathcal{P}_H^t} E^{\mathbb{P}^t})$, since it is a probability of an intersection of non-countable sets. Therefore, the answer to the first question is negative.

For the same reason, the answer to the second question is negative either, unless we could find an α such that for each $a \in \mathbb{S}_d^{>0}$, $\alpha^{t,\omega}(a)$ is independent of ω, i.e., $\alpha_s^{t,\omega}(a) \equiv \alpha_s^t(a)$. In such case, if we only assume the second point and define

$$E^{\mathbb{P}^t} := \left\{\omega : \int_t^1 \alpha_s^{t,\omega}(\hat{a}_s^t) ds \le \overline{\alpha},\ \mathbb{P}^t - a.s.\right\},$$

then $E^{\mathbb{P}^t} = \Omega$ for all $\mathbb{P}^t \in \mathcal{P}_H^t$, which implies the third point in (A3"). As we have shown in (A3'), a special case of such α is that for each $a \in \mathbb{S}_d^{>0}$, $\alpha_\cdot(a)$ is a deterministic function in t.

Remark 4.35 *Corresponding to (v) of Assumption 2.2 in Possamaï and Zhou [78], (A5) can be weakened to the following form:*

(A5") *F is local Lipschitz in z, i.e., for each $(\omega, t, y, z, z', a) \in \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times D_{F_t}$,*

$$|F_t(\omega, y, z, a) - F_t(\omega, y, z', a)| \le C(|a^{1/2}\phi_t(a)| + |a^{1/2} z| + |a^{1/2} z'|)|a^{1/2}(z - z')|,$$

where C is a strictly positive constant and ϕ satisfies that

- *For each $a \in \mathbb{S}_d^{>0}$, $\phi(a)$ is an $\mathcal{F}$-progressively measurable process;*
- *$\phi(\hat{a})$ is a $BMO(\mathcal{P}_H)$-martingale generator;*
- *For each $(\omega, t) \in \Omega \times (0,1]$, $\phi^{t,\omega}(\hat{a}^t)$ is a $BMO(\mathbb{P}^t)$-martingale generator under each $\mathbb{P}^t \in \mathcal{P}_H^t$.*

Based on the argument in remark 4.34, only having that ϕ is a $BMO(\mathcal{P}_H)$-martingale generator, we have no idea wether for each $(\omega, t) \in \Omega \times [0,1]$, $\phi^{t,\omega}$ is a $BMO(\mathbb{P}^t)$-martingale generator under all $\mathbb{P}^t \in \mathcal{P}_H^t$, unless for each $a \in \mathbb{S}_d^{>0}$, $\phi(a)$ is independent of ω. Thus, the third point in (A5") is necessary. We would like to point out that (v) in Assumption 2.2 in Possamaï and Zhou [78] causes slight problems for their setting of $V_t(\omega)$ and for the proof of Lemma 5.1 in that paper.

Taking the 2BSDE (4.42) as an example, we explain these settings ((A3') and (A5') are special cases of (A3") and (A5"), respectively). We observe that the generator (4.43) satisfies the quadratic condition (A3") for

$$\alpha_t(a) := 2c\inf\{|r|^2 : r \in a^{1/2}\tilde{C}\} + \frac{5+c}{2c}\mathrm{tr}(a^{-1})M^2,\ a \in \mathbb{S}_d^{>0},$$

in which $\alpha(a)$ is a deterministic function. In general, $\inf\{|r|^2 : r \in a^{1/2}\tilde{C}\}$ and $|a^{-1/2}b|^2$ could be unbounded, so that (A3) is no longer satisfied. If we choose $\overline{\alpha} = 2c\overline{K}^2 + \frac{5+c}{2c}\underline{K}^2$, then (A3") is satisfied.

Similarly, the generator (4.43) satisfies no longer (A5). We define

$$\phi_t(a) := 2\inf\{|r| : r \in a^{1/2}\tilde{C}\} + \frac{4}{c}(\mathrm{tr}(a^{-1}))^{1/2}M,\ a \in \mathbb{S}_d^{>0},$$

which is bounded by $2\overline{K} + \frac{4}{c}\underline{K}$ when a is replaced by $\hat{a}$ (or $\hat{a}^t$, respectively), $\tilde{\mathcal{P}}_H$ (or $\tilde{\mathcal{P}}_H^t$, respectively)-q.s.. By Definition 4.24, we know that a constant process is a $BMO(\tilde{\mathcal{P}}_H)$ (or $BMO(\tilde{\mathcal{P}}_H^t)$, respectively)-martingale generator. Then, (A5") is satisfied.

The wellposedness of 2BSDEs will not alter under (A3") and (A5"). First, the statement (4.7) remains true if we change a little of its expression:

$$E_\tau^{\mathbb{P}}\left[\int_\tau^1 |\hat{a}_t^{1/2}Z_t|^2\right] \leq \frac{1}{\gamma^2}e^{4\gamma||Y||_{\mathbb{D}_H^\infty}}(1 + 2\gamma(\overline{\alpha} + \overline{\beta}||Y||_{\mathbb{D}_H^\infty})),$$

which yields that Z is a $BMO(\mathcal{P}_H)$-martingale generator if $Y \in \mathbb{D}_H^\infty$. Lemma 4.6 and 4.7 ensure that the constants that we need for the proof of the representation theorem and the last step of the proof to the existence are uniform in $\mathbb{P}$. For the existence result, we have already explained that $V_t(\omega)$ in (4.28) is well defined and all properties still hold since (A3") and (A5") provide existence and uniqueness results as well as estimates of solutions to quadratic BSDEs with parameters $(\xi^{t,\omega}, \hat{F}^{t,\omega})$ under each $\mathbb{P}^t \in \mathcal{P}_H^t$.

Remark 4.36 *If we assume in addition that* $0 \in \tilde{C}$, *then* $\overline{K} = \inf\{|r| : r \in a^{1/2}\tilde{C}\} = 0$, *so that the upper bound of* $\hat{a}$ *is not necessary to be uniform. Both Theorem 4.27 and 4.31 can hold true under a larger class of probability measures* $\hat{\mathcal{P}}_H$*:*

Definition 4.37 *We denote by* $\hat{\mathcal{P}}_H$ *the collection of all those* $\mathbb{P} \in \overline{P}_S$ *such that*

$$\overline{a}^{\mathbb{P}} \leq \hat{a}_t \leq \underline{a}^{\mathbb{P}},\ \mathrm{tr}(\hat{a}_t^{-1}) \leq \alpha_t,\ \text{and}\ \hat{a}_t \in D_{F_t},\ \lambda \times \mathbb{P} - a.e.,$$

for some $\overline{a}^{\mathbb{P}}, \underline{a}^{\mathbb{P}} \in \mathbb{S}_d^{>0}$, *a strictly positive* $\alpha \in L^1([0,1])$ *and each* $(y,z) \in \mathbb{R} \times \mathbb{R}^d$.

Correspondingly, we denote by $\hat{\mathcal{P}}_H^t$ *the collection of all those* $\mathbb{P}^t \in \overline{P}_S^t$ *such that*

$$\overline{a}^{\mathbb{P}^t} \leq \hat{a}_s^t \leq \underline{a}^{\mathbb{P}^t},\ \mathrm{tr}((\hat{a}_s^t)^{-1}) \leq \alpha_s\ \text{and}\ \hat{a}_s^t \in D_{F_s},\ \lambda \times \mathbb{P}^t - a.e.,$$

for some $\overline{a}^{\mathbb{P}^t}, \underline{a}^{\mathbb{P}^t} \in \mathbb{S}_d^{>0}$, *the same* α *as above and each* $(y,z) \in \mathbb{R} \times \mathbb{R}^d$.

We can verify that this series $\{\mathcal{P}_H^t\}_{t\in[0,1]}$ *defined by Definition 4.37 is consistent and they ensure that (4.43) and (4.55) satisfy (A3") and (A5"), respectively.*

However, if we consider the same problems under an even larger class of probability measures $\breve{\mathcal{P}}_H$*:*

Definition 4.38 *We denote by* $\breve{\mathcal{P}}_H$ *the collection of all* $\mathbb{P} \in \overline{P}_S$ *such that*

$$\overline{a}^{\mathbb{P}} \leq \hat{a}_t \leq \underline{a}^{\mathbb{P}},\ \int_0^1 \mathrm{tr}(\hat{a}_t^{-1})dt \leq \overline{\alpha}\ \text{and}\ \hat{a}_t \in D_{F_t},\ \lambda \times \mathbb{P} - a.e.,$$

for some $\overline{a}^{\mathbb{P}}, \underline{a}^{\mathbb{P}} \in \mathbb{S}_d^{>0}$, *some strictly positive constant* $\overline{\alpha}$ *and each* $(y,z) \in \mathbb{R} \times \mathbb{R}^d$.

then the wellposedness of (4.42) and (4.54) will no longer hold true, since one is difficult to find a series of class $\{\breve{\mathcal{P}}_H^t\}_{t\in[0,1]}$ *consistent with* $\breve{\mathcal{P}}_H$ *defined by Definition 4.38. In another word, once* $\breve{\mathcal{P}}_H^t$ *contains all the r.c.p.d.* $\mathbb{P}^{t,\omega}$ *of* $\mathbb{P} \in \breve{\mathcal{P}}_H$, *the second point in Definition 4.33 could not hold true.*

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Résumé

Équations différentielles stochastiques sous les espérances mathématiques non-linéaires et applications

Cette thèse est composée de deux parties indépendantes : la première partie traite des équations différentielles stochastiques dans le cadre de la G-espérance, tandis que la deuxième partie présente les résultats obtenus pour les équations différentielles stochastiques du seconde ordre.

Dans un premier temps, on considère les intégrales stochastiques par rapport à un processus croissant, et on donne une extension de la formule d'Itô dans le cadre de la G-espérance. Ensuite, on étudie une classe d'équations différentielles stochastiques réfléchies unidimensionnelles dirigées par un G-mouvement brownien. Dans la suite, en utilisant une méthode de localisation, on prouve l'existence et l'unicité de solutions pour les équations différentielles stochastiques dirigées par un G-mouvement brownien, dont les coefficients sont localement lipschitziens. Enfin, dans le même cadre, on discute des problèmes de réflexion multidimensionnelle et on fournit quelques résultats de convergence.

Dans un deuxième temps, on étudie une classe d'équations différentielles stochastiques rétrogrades du seconde ordre à croissance quadratique. Le but de ce travail est de généraliser le résultat obtenu par Possamaï et Zhou en 2012. On montre aussi l'existence et l'unicité des solutions pour ces équations, mais sous des hypothèses plus faibles. De plus, ce résultat théorique est appliqué aux problèmes de maximisation robuste de l'utilité du portefeuille en finance.

Mots clés : G-mouvement brownien ; équations différentielles stochastiques ; frontières de réflexion ; temps d'arrêts ; équations différentielles stochastiques rétrogrades du seconde ordre ; croissance quadratique ; maximisation robuste de l'utilité.

Abstract

Stochastic Differential Equations under Nonlinear Mathematical Expectations and their Applications

This thesis consists of two relatively independent parts : the first part concerns stochastic differential equations in the framework of the G-expectation, while the second part deals with a class of second order backward stochastic differential equations.

In the first part, we first consider stochastic integrals with respect to an increasing process and give an extension of Itô's formula in the G-framework. Then, we study a class of scalar valued reflected stochastic differential equations driven by G-Brownian motion. Subsequently, we prove the existence and the uniqueness of solutions for some locally Lipschitz stochastic differential equations driven by G-Brownian motion. At the end of this part, we consider multidimensional reflected problems in the G-framework, and some convergence results are obtained.

In the second part, we study the wellposedness of a class of second order backward stochastic differential equations (2BSDEs) under a quadratic growth condition on their coefficients. The aim of this part is to generalize a wellposedness result for quadratic 2BSDEs by Possamaï and Zhou in 2012. In this thesis, we work under some usual assumptions and deduce the existence and uniqueness theorem as well. Moreover, this theoretical result for quadratic 2BSDEs is applied to solve some robust utility maximization problems in finance.

Keywords : G-Brownian motion ; stochastic differential equations ; reflecting boundary ; stopping times ; second order backward stochastic differential equations ; quadratic growth ; robust utility maximization.